Economics 202N: Core Economics 1 and 2

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Part I

“Comfortable” Mathematics

Outline

- Algebra
- Calculus
- Linear algebra
- Set theory
- Probability
- Problem-solving and proof

Solving single and systems of equations I

$$x^2 + x - 6 = 0$$

Solving single and systems of equations II

$$a + b = 4$$
$$a^2 + b = 10$$

Solving single and systems of equations III

$$\frac{x^2 - 9}{x - 3}$$

Exponentiation and logarithms I

$$\log(\gamma^3)$$

Exponentiation and logarithms II

$$e^{\log(4t)}$$

Graphs and functional forms I

$$y = mx + b$$

Graphs and functional forms II

$$y = \frac{1}{x^2}$$

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Single-variable derivation I

$$\frac{d}{dx}7x^3$$

Single-variable derivation II

$$\frac{d}{d\alpha} e^{(\alpha^2)}$$

Integration

$$\int x^2 \, dx$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Partial derivation

$$f(x,y) = \frac{\log x}{y} \implies \begin{aligned} \frac{\partial f}{\partial x} &= \\ \frac{\partial f}{\partial y} &= \\ \frac{\partial^2 f}{\partial x \, \partial y} &= \end{aligned}$$

Maximization and minimization

$$f(\xi) = \xi^2 - 8\xi + 13\sqrt{2}$$

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Matrix multiplication I

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Matrix multiplication II

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -3 \end{bmatrix}$$

Determinants

$$\det \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

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Set notation I

$$A \subseteq B$$

Set notation II

$$\exists b \in B \colon f(b) = 3$$

Set logic

$$A \subseteq B$$

$$\exists b \in B: f(b) = 3$$

$$a \in A \implies f(a) =$$

Outline

- Algebra
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- Probability
- Problem-solving and proof

Probability concepts

- Probability mass/density functions
- Cumulative distribution functions
- Expected value
- Jensen's inequality

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"Word problems"

Suppose apples cost p_a and bananas cost p_b. If I have d dollars and buy a apples, how many bananas can I afford?

Proof techniques

- Counterexample
- Exhaustion
- Contradiction
- Induction
- ...

Part II

Producer Theory

Individual decision-making under certainty

Objects of inquiry

Our study of microeconomics begins with individual decision-making under certainty

Items of interest include:

- Feasible set
- Objective function (Feasible set $\to \mathbb{R}$)
- Choice correspondence (Parameters $\rightrightarrows$ Feasible set)
- "Maximized" objective function (Parameters $\to \mathbb{R}$)

Individual decision-making under certainty

Course outline

We will divide decision-making under certainty into three units:

1. Producer theory
   - Feasible set defined by technology
   - Objective function $p \cdot y$ depends on prices
2. Abstract choice theory
   - Feasible set totally general
   - Objective function may not even exist
3. Consumer theory
   - Feasible set defined by budget constraint and depends on prices
   - Objective function $u(x)$

Producer theory: simplifying assumptions

Standard model: firms choose production plans (technologically feasible lists of inputs and outputs) to maximize profits

Simplifying assumptions include:

1. Firms are price takers (both input and output markets)
2. Technology is exogenously given
3. Firms maximize profits; should be true as long as
   - The firm is competitive
   - There is no uncertainty about profits
   - Managers are perfectly controlled by owners

Role of simplifying assumptions

No consensus about "correct" view

Modeling is an abstraction

- Relies on simplifying but untrue assumptions
- Highlight important effects by suppressing other effects
- Basis for numerical calculations

Models can be useful in different ways

- Relevant predictions reasonably accurate; can sometimes be checked using data or theoretical analysis
- Failure of relevant predictions can highlight which simplifying assumptions are most relevant
- "Usual" or "standard" models often fail realism checks; do not skip validation

Outline

- Production sets
- Profit maximization
- Rationalizability
- Rationalizability: the differentiable case
- Single-output firms

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Production sets

Exogenously given technology applies over n commodities (both inputs and outputs)

Definition (production plan)

A vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ where an output has $y_k > 0$ and an input has $y_k < 0$.

Definition (production set)

Set $Y \subseteq \mathbb{R}^n$ of feasible production plans; generally assumed to be non-empty and closed.

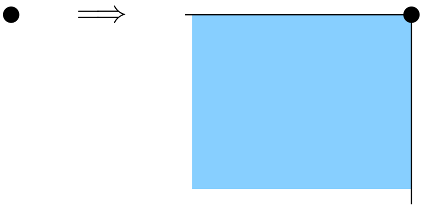
Properties of production sets I

Definition (shutdown)

$\mathbf{0} \in Y$.

Definition (free disposal)

$y \in Y$ and $y' \leq y$ imply $y' \in Y$.



Properties of production sets II

Definition (nonincreasing returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0, 1]$.

Implies shutdown

Definition (nondecreasing returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geq 1$.

Along with shutdown, implies $\pi(p) = 0$ or $\pi(p) = +\infty$ for all p

Definition (constant returns to scale)

$y \in Y$ implies $\alpha y \in Y$ for all $\alpha \geq 0$; i.e., nonincreasing *and* nondecreasing returns to scale.

Properties of production sets III

Definition (convex production set)

$y, y' \in Y$ imply $ty + (1-t)y' \in Y$ for all $t \in [0, 1]$.

Vaguely "nonincreasing returns to specialization"
If $\mathbf{0} \in Y$, then convexity implies nonincreasing returns to scale

Strictly convex iff for $t \in (0, 1)$, the convex combination is in the interior of Y

Characterizing Y: Transformation function I

Definition (transformation function)

Any function $T \colon \mathbb{R}^n \to \mathbb{R}$ with

1. $T(y) \leq 0 \iff y \in Y$; and
2. $T(y) = 0 \iff y$ is a boundary point of Y.

Can be interpreted as the amount of technical progress required to make y feasible

The set $\{y \colon T(y) = 0\}$ is the production possibilities frontier (a.k.a. transformation frontier)

Characterizing Y: Transformation function II

When the transformation function is differentiable, we can define the marginal rate of transformation of good l for good k:

Definition (marginal rate of transformation)

$$\text{MRT}_{l,k}(y) \equiv \frac{\frac{\partial T(y)}{\partial y_l}}{\frac{\partial T(y)}{\partial y_k}},$$

defined for points where $T(y) = 0$ and $\frac{\partial T(y)}{\partial y_k} \neq 0$.

Measures the extra amount of good k that can be obtained per unit reduction of good l

Equals the slope of the PPF

Outline

- Production sets
- Profit maximization
- Rationalizability
- Rationalizability: the differentiable case
- Single-output firms

The Profit Maximization Problem

The firm's optimal production decisions are given by correspondence $y \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

$$y(p) \equiv \operatorname*{argmax}_{y \in Y} p \cdot y$$
$$= \{y \in Y : p \cdot y = \pi(p)\}$$

Resulting profits are given by profit function $\pi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$

$$\pi(p) \equiv \sup_{y \in Y} p \cdot y$$

A note on maxima and suprema

We have a tendency to be fast and loose with these, but recall that:

- A maximum is the highest achieved value
- A supremum is a least upper bound (which may or may not be achieved)

Fact

We have not made sufficient assumptions to ensure that a maximum profit is achieved (i.e., $y(p) \neq \varnothing$), and so the sup *cannot necessarily be replaced with a* max.
In particular we allow for the possibility that $\pi(p) = +\infty$, which can happen if Y is unbounded.

A note on convex functions

Definition (convexity)

$f : \mathbb{R}^n \to \mathbb{R}$ is convex iff for all x and $y \in \mathbb{R}^n$, and all $\lambda \in [0,1]$, we have

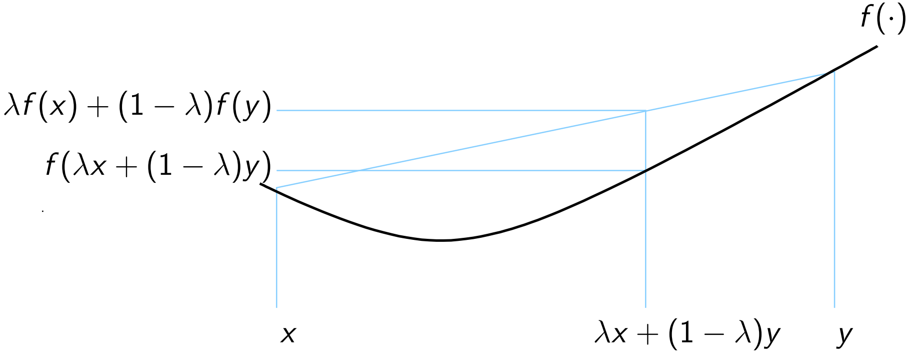
$$\lambda f(x) + (1-\lambda) f(y) \geq f\big(\lambda x + (1-\lambda) y\big).$$

In the differentiable case, also characterized by any of

- If $f : \mathbb{R} \to \mathbb{R}$, then $f''(x) \geq 0$ for all x
- Hessian $\nabla^2 f(x)$ is a positive semidefinite matrix for all x
- $f(\cdot)$ lies above its tangent hyperplanes; i.e.,

$$f(x) \geq f(y) + \nabla f(y) \cdot (x - y) \text{ for all } x \text{ and } y$$

Convex function illustration



Convexity of $\pi(\cdot)$

Theorem

$\pi(\cdot)$ is a convex function.

Proof.

Fix any p_1, p_2 and let $p_t \equiv tp_1 + (1-t)p_2$ for $t \in [0,1]$. Then for any $y \in Y$,

$$\begin{aligned} p_t \cdot y &= t \underbrace{p_1 \cdot y}_{\leq \pi(p_1)} + (1-t) \underbrace{p_2 \cdot y}_{\leq \pi(p_2)} \\ &\leq t\pi(p_1) + (1-t)\pi(p_2). \end{aligned}$$

Since this is true for *all* $p_t \cdot y$, it holds for $\sup_{y \in Y} p_t \cdot y = \pi(p_t)$:

$$\pi(p_t) \leq t\pi(p_1) + (1-t)\pi(p_2).$$

□

A note on homogeneous functions

Definition (homogeneity)

$f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k iff for all $x \in \mathbb{R}^n$, and all $\lambda > 0$, we have

$$f(\lambda x) = \lambda^k f(x).$$

We will overwhelmingly rely on

- Homogeneity of degree zero: $f(\lambda x) = f(x)$
- Homogeneity of degree one: $f(\lambda x) = \lambda f(x)$

Euler's Law I

Theorem (Euler's Law)

Suppose $f(\cdot)$ is differentiable. Then it is homogeneous of degree k iff $p \cdot \nabla f(p) = kf(p)$.

Proof.

Homogeneous $\Rightarrow p \cdot \nabla f(p) = kf(p)$ proved by differentiating $f(\lambda p) = \lambda^k f(p)$ with respect to λ, and then setting $\lambda = 1$.

Homogeneous $\Leftarrow p \cdot \nabla f(p) = kf(p)$ may be covered in section. □

Euler's Law II

Corollary

If $f(\cdot)$ is homogeneous of degree one, then $\nabla f(\cdot)$ is homogeneous of degree zero.

Proof.

Homogeneity of degree one means

$$\lambda f(p) = f(\lambda p).$$

Differentiating in p,

$$\begin{aligned} \lambda \nabla f(p) &= \lambda \nabla f(\lambda p) \\ \nabla f(p) &= \nabla f(\lambda p) \end{aligned}$$

□

Homogeneity of $\pi(\cdot)$

Theorem

$\pi(\cdot)$ is homogeneous of degree one; i.e., $\pi(\lambda p) = \lambda\pi(p)$ for all p and $\lambda > 0$.

That is, if you scale all (input and output) prices up or down the same amount, you also scale profits by that amount

Proof.

$$\begin{aligned}\pi(\lambda p) \equiv & \sup_{y \in Y} \lambda p \cdot y \\ &= \lambda \sup_{y \in Y} p \cdot y \\ &= \lambda \pi(p).\end{aligned}$$

□

Homogeneity of $y(\cdot)$

Theorem

$y(\cdot)$ is homogeneous of degree zero; i.e., $y(\lambda p) = y(p)$ for all p and $\lambda > 0$.

That is, a firm makes the same production choice if all (input and output) prices are scaled up or down the same amount

Proof.

$$
\begin{aligned}
y(\lambda p) &\equiv \{y \in Y : \lambda p \cdot y = \pi(\lambda p)\} \\
&= \{y \in Y : \lambda p \cdot y = \lambda \pi(p)\} \\
&= \{y \in Y : \quad p \cdot y = \quad \pi(p)\} \\
&= y(p).
\end{aligned}
$$

□

Outline

- Production sets
- Profit maximization
- **Rationalizability**
- Rationalizability: the differentiable case
- Single-output firms

Recovering the feasible set

Rationalizability asks for a given $y(\cdot)$ and/or $\pi(\cdot)$—which we may not observe everywhere—about properties of the underlying Y.

Suppose that we don't know Y, but observe *some* supply decisions $\tilde{y}(p) \subseteq y(p)$ and/or resulting profits $\tilde{\pi}(p) = \pi(p)$ when it faces price vectors p from a set $P \subseteq \mathbb{R}^n$

1. What can we infer about the underlying production set Y?
2. Is there *any* Y such that $\tilde{y}(p)$ and $\pi(p)$ are consistent with profit maximization?
3. Can we recover the entire production set if we have "enough data"?

Rationalizability: definitions

Definitions (rationalization)

- Supply correspondence $\tilde{y}\colon P \rightrightarrows \mathbb{R}^n$ is rationalized by production set Y iff $\forall p \in P$, $\tilde{y}(p) \subseteq \operatorname{argmax}_{y \in Y} p \cdot y$.
- Profit function $\tilde{\pi}\colon P \to \mathbb{R} \cup \{+\infty\}$ is rationalized by production set Y iff $\forall p$, $\tilde{\pi}(p) = \sup_{y \in Y} p \cdot y$.

Definitions (rationalizability)

- $\tilde{y}(\cdot)$ or $\tilde{\pi}(\cdot)$ is rationalizable if it is rationalized by *some* production set.
- $\tilde{y}(\cdot)$ and $\tilde{\pi}(\cdot)$ are jointly rationalizable if they are both rationalized by the *same* production set.

When are $\pi(\cdot)$ and $y(\cdot)$ jointly rationalized by Y? I

Question 1

What can we infer about the underlying production set Y?

1. Production plans the firm actually chooses must be feasible
   - The set of chosen production plans gives an "inner bound"

$$Y^I \equiv \bigcup_{p \in P} \tilde{y}(p)$$

   - If $\tilde{y}(\cdot)$ is rationalized by Y, we must have $Y^I \subseteq Y$

When are $\pi(\cdot)$ and $y(\cdot)$ jointly rationalized by Y? II

2. Production plans that yield *higher* profits than those chosen must be infeasible
   - The set of production plans less profitable than $\tilde{\pi}(p)$ at price p gives an "outer bound"

$$\begin{aligned} Y^O &\equiv \{y \colon p \cdot y \leq \tilde{\pi}(p) \text{ for all } p \in P\} \\ &\equiv \{y \colon p \cdot y \leq p \cdot \tilde{y}(p) \text{ for all } p \in P\} \end{aligned}$$

   - If $\tilde{\pi}(\cdot)$ is rationalized by Y, we must have $Y \subseteq Y^O$

When are $\pi(\cdot)$ and $y(\cdot)$ jointly rationalized by Y? III

Theorem

A nonempty-valued supply correspondence $\tilde{y}(\cdot)$ and profit function $\tilde{\pi}(\cdot)$ on a price set are jointly rationalized by production set Y iff

1. $p \cdot y = \tilde{\pi}(p)$ *for all* $y \in \tilde{y}(p)$ *(adding-up), and*
2. $Y^I \subseteq Y \subseteq Y^O$.

Proof.

Rationalized by $Y \Rightarrow$ conditions by construction of Y^I and Y^O as argued above.

Rationalized by $Y \Leftarrow$ conditions since for any price vector p, the firm can achieve profit $\tilde{\pi}(p)$ by choosing any $y \in \tilde{y}(p) \subseteq Y^I \subseteq Y$, but cannot achieve any higher profit since $Y \subseteq Y^O$. □

When are $\pi(\cdot)$ and $y(\cdot)$ jointly rationalizable?

Question 2

Which observations are rationalizable, i.e., consistent with profit maximization for *some* production set?

Corollary

A nonempty-valued supply correspondence $\tilde{y}(\cdot)$ and profit function $\tilde{\pi}(\cdot)$ on a price set are jointly rationalizable iff

1. $p \cdot y = \tilde{\pi}(p)$ *for all* $y \in \tilde{y}(p)$ *(adding-up), and*
2. $Y^I \subseteq Y^O$*; i.e.,* $p \cdot y' \leq \tilde{\pi}(p)$ *for all* p, p'*, and all* $y' \in \tilde{y}(p')$ *(Weak Axiom of Profit Maximization).*

Fully recovering Y from $\pi(\cdot)$ and $y(\cdot)$ I

Question 3

Can we recover the entire production set if we have enough data?

Theorem

Suppose we observe profits $\pi(\cdot)$ for all nonnegative prices ($P = \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$), and further assume

1. *Y satisfies free disposal, and*
2. *Y is convex and closed.*

Then $Y = Y^O$.

Fully recovering Y from $\pi(\cdot)$ and $y(\cdot)$ II

Why do we need convexity and closure of Y?

- Closure makes it "more likely" that $\pi(p)$ is actually achieved (i.e., the supremum is also the maximum)
- Convexity is a bit trickier...
  - The outer bound is defined as the intersection of linear half-spaces

$$\begin{aligned} Y^O &\equiv \{y \colon p \cdot y \leq \pi(p) \text{ for all } p \in P\} \\ &= \bigcap_{p \in P} \{y \colon p \cdot y \leq \pi(p)\} \end{aligned}$$

  - Thus Y^O is convex (since it is the intersection of convex sets)

A note on the Separating Hyperplane Theorem I

Theorem (Separating Hyperplane Theorem)

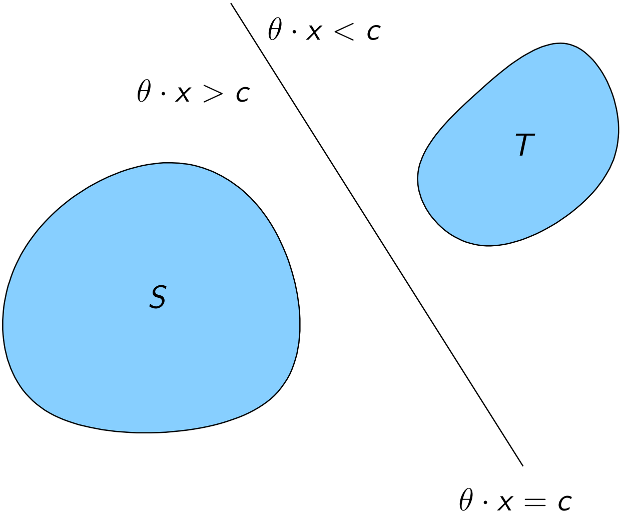
Suppose that S and T are two convex, closed, and disjoint ($S \cap T = \varnothing$) subsets of $\mathbb{R}^n$. Then there exists $\theta \in \mathbb{R}^n$ and $c \in \mathbb{R}$ with $\theta \neq \mathbf{0}$ such that

$$\theta \cdot s \geq c \text{ for all } s \in S \text{ and } \theta \cdot t < c \text{ for all } t \in T.$$

Means that a convex, closed set can be separated from any point outside the set

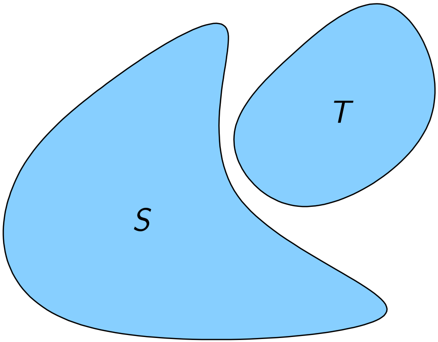
SHT is one of a few key tools for proving many of our results

A note on the Separating Hyperplane Theorem II



A note on the Separating Hyperplane Theorem III

We can't necessarily separate nonconvex sets:



Fully recovering Y from $\pi(\cdot)$ and $y(\cdot)$ reprise I

Question 3

Can we recover the entire production set if we have enough data?

Theorem

Suppose we observe profits $\pi(\cdot)$ for all nonnegative prices ($P = \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$), and further assume

1. *Y satisfies free disposal, and*
2. *Y is convex and closed.*

Then $Y = Y^O$.

Fully recovering Y from $\pi(\cdot)$ and $y(\cdot)$ reprise II

Proof.

We know $Y \subseteq Y^O$; thus we only need to show that $Y^O \subseteq Y$.

Take any $x \notin Y$. Y and $\{x\}$ are closed, convex, and disjoint, so we can apply the Separating Hyperplane Theorem: there exists $p \neq \mathbf{0}$ such that $p \cdot x > \sup_{y \in Y} p \cdot y = \pi(p)$.

By free disposal, if any component of p were negative, then $\sup_{y \in Y} p \cdot y = +\infty$. So $p > \mathbf{0}$; i.e., $p \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\} = P$. But since $p \cdot x > \pi(p)$, it must be that $x \notin Y^O$.

We have showed that $x \notin Y \Rightarrow x \notin Y^O$, or equivalently $Y^O \subseteq Y$. □

Outline

- Production sets
- Profit maximization
- Rationalizability
- Rationalizability: the differentiable case
- Single-output firms

Loss function

We can also describe the feasible set using a "loss function"

Definition (loss function)

$L(p, y) \equiv \pi(p) - p \cdot y$. This is the loss from choosing y rather than the profit-maximizing feasible production plan.

If $L(p, y) < 0$, then $p \cdot y > \pi(p)$, and hence y must be infeasible

The outer bound can therefore be written

$$\begin{aligned} Y^O &\equiv \{y : p \cdot y \leq \pi(p) \text{ for all } p \in P\} \\ &= \{y : \inf_{p \in P} L(p, y) \geq 0\}, \end{aligned}$$

i.e., the set of points at which losses are nonnegative at any price

Hotelling's Lemma I

Assume rationalizability

Consider any $p' \in P$, and any $y' \in y(p')$:

- $y' \in Y^I$ (by definition)
- Thus $y' \in Y^O = \{y\colon \inf_{p \in P} L(p, y) \geq 0\}$ (by WAPM)
- That is, $\inf_{p \in P} L(p, y') \geq 0$
- But by adding-up, $p' \cdot y' = \pi(p')$, so $L(p', y') = 0$
- Thus the infimum is achieved, and equals the minimum:

$$\min_{p \in P} L(p, y') = L(p', y') = 0 \text{ for all } p' \in P \text{ and } y' \in y(p')$$

Losses from making production choice y' at price p when the actual price is p' must be nonnegative, and are exactly zero when $p = p'$

Hotelling's Lemma II

Dual problem: loss minimization

The loss minimization problem $\min_{p \in P} L(p, y')$ for $L(p, y) \equiv \pi(p) - p \cdot y$ is solved at $p = p'$ whenever $y' \in y(p')$:

$$\min_{p \in P} L(p, y') = L(p', y') = 0.$$

We can apply a first-order condition since

- The set P is open, so all its points are interior
- At any point at which $\pi(\cdot)$ is differentiable, so is $L(\cdot, y')$

Hotelling's Lemma III

This FOC is

Theorem (Hotelling's Lemma)

$$\nabla_p L(p, y')\big|_{p=p'} = 0 \textit{ for all } y' \in y(p').$$

Dispensing with the loss function gives $\nabla \pi(p') = y'$.

This can also be viewed as an application of the Envelope Theorem to the Profit Maximization Problem: $\pi(p) = \sup_{y \in Y} p \cdot y$

- ETs relate the derivatives of the objective and value functions

Implications of Hotelling's Lemma

Recall

Theorem (Hotelling's Lemma)

$\nabla\pi(p) = y(p)$ *wherever* $\pi(\cdot)$ *is differentiable.*

- Thus if $\pi(\cdot)$ is differentiable at p, $y(p)$ is a singleton
- We restrict ourselves to this case; we can call $y(\cdot)$ a supply *function* rather than the more general supply *correspondence*
- The notes include a section on the nondifferentiable case, which we are going to skip

Rationalization: $y(\cdot)$ and differentiable $\pi(\cdot)$ I

Theorem

$y \colon P \to \mathbb{R}^n$ (the correspondence ensured to be a function by Hotelling's lemma, given differentiable $\pi(\cdot)$) and differentiable $\pi \colon P \to \mathbb{R}$ on an open convex set $P \subseteq \mathbb{R}^n$ are jointly rationalizable iff

1. $p \cdot y(p) = \pi(p)$ (*adding-up*),
2. $\nabla \pi(p) = y(p)$ (*Hotelling's Lemma*), *and*
3. $\pi(\cdot)$ *is convex.*

Note that

- Condition 2 describes the first-order condition and
- Condition 3 describes the second-order condition

of the dual (loss minimization) problem

Rationalization: $y(\cdot)$ and differentiable $\pi(\cdot)$ II

Proof.

We showed earlier that 2 and 3 follow from rationalizability.

It remains to be shown that 1–3 imply WAPM (i.e., $\pi(p) \geq p \cdot y(p'))$.

Noting that a convex function lies above its tangent hyperplanes, and applying Hotelling's Lemma and adding-up gives

$$\begin{aligned}\pi(p) &\geq \pi(p') + (p - p') \cdot \nabla\pi(p') \\ &= \pi(p') + (p - p') \cdot y(p') \\ &= p' \cdot y(p') + (p - p') \cdot y(p') \\ &= p \cdot y(p').\end{aligned}$$

□

Rationalization: differentiable $y(\cdot)$ I

Theorem

Differentiable $y\colon P \to \mathbb{R}^n$ on an open convex set $P \subseteq \mathbb{R}^n$ is rationalizable iff

1. *$y(\cdot)$ is homogeneous of degree zero, and*
2. *The Jacobian $Dy(p)$ is symmetric and positive semidefinite.*

Rationalization: differentiable $y(\cdot)$ II

Proof.

We showed earlier that if $y(\cdot)$ is rationalizable, it

1. Is homogeneous of degree zero; and
2. Satisfies Hotelling's Lemma, thus $Dy(p) = D^2\pi(p)$ is symmetric PSD (it is the Hessian of a convex function).

Now suppose conditions of the theorem hold. Take $\pi(p) = p \cdot y(p)$. For each $i = 1, \dots, n$,

$$\frac{\partial \pi(p)}{\partial p_i} = y_i(p) + \sum_j p_j \frac{\partial y_j(p)}{\partial p_i} = y_i(p) + \underbrace{\sum_j p_j \frac{\partial y_i(p)}{\partial p_j}}_{= p \cdot \nabla y_i(p) = 0} = y_i(p)$$

Thus $D^2\pi(p) = Dy(p)$ is PSD, hence $\pi(\cdot)$ is convex. Thus $y(\cdot)$ and $\pi(\cdot)$ are jointly rationalizable. □

Rationalization: differentiable $\pi(\cdot)$

Theorem

Differentiable $\pi : P \to \mathbb{R}$ on a convex set $P \subseteq \mathbb{R}^n$ is rationalizable iff

1. *$\pi(\cdot)$ is homogeneous of degree one, and*
2. *$\pi(\cdot)$ is convex.*

Proof.

We showed earlier that if $\pi(\cdot)$ is rationalizable, it is homogeneous of degree one and convex.
Now suppose conditions of the theorem hold. Take $y(p) = \nabla\pi(p)$. By Euler's Law, $\pi(p) = p \cdot \nabla\pi(p) = p \cdot y(p)$. Thus $y(\cdot)$ and $\pi(\cdot)$ are jointly rationalizable. □

Substitution matrix

Definition (substitution matrix)

The Jacobian of the optimal supply function,

$$Dy(p) \equiv \left[\frac{\partial y_i(p)}{\partial p_j}\right]_{i,j} \equiv \begin{bmatrix} \frac{\partial y_1(p)}{\partial p_1} & \cdots & \frac{\partial y_1(p)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n(p)}{\partial p_1} & \cdots & \frac{\partial y_n(p)}{\partial p_n} \end{bmatrix}.$$

- By Hotelling's Lemma, $Dy(p) = D^2\pi(p)$, hence the substitution matrix is symmetric
  - A "subtle conclusion of mathematical economics"
- Convexity of $\pi(\cdot)$ implies positive semidefiniteness
  - Supply curves must be upward sloping (the "Law of Supply")

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The single-output firm: notation

Notation will be a bit different for single-output firms:

$p \in \mathbb{R}_+$: Price of output

$w \in \mathbb{R}_+^{n-1}$: Prices of inputs

$q \in \mathbb{R}_+$: Output produced

$z \in \mathbb{R}_+^{n-1}$: Inputs used

Thus $p_{\text{old}} = (p, w)$ and $y_{\text{old}} = (q, -z)$
We will often label $m \equiv n - 1$

Characterizing Y: Production function I

Definition (production function)

For a firm with only a single output q (and inputs $-z$), defined as $f(z) \equiv \max q$ such that $(q, -z) \in Y$.

$Y = \{(q, -z) \colon q \leq f(z)\}$, assuming free disposal

Characterizing Y: Production function II

When the production function is differentiable, we can define the marginal rate of technological substitution of good l for good k:

Definition (marginal rate of technological substitution)

$$\text{MRTS}_{l,k}(z) \equiv \frac{\frac{\partial f(z)}{\partial z_l}}{\frac{\partial f(z)}{\partial z_k}},$$

defined for points where $\frac{\partial f(z)}{\partial z_k} \neq 0$.

Measures how much of input k must be used in place of one unit of input l to maintain the same level of output

Dividing up the problem I

With one output, free disposal, and production function $f(\cdot)$,

$$Y = \{(q, -z) \colon z \in \mathbb{R}^m_+ \text{ and } f(z) \geq q\}$$

Given a positive output price $p > 0$, profit maximization requires $q = f(z)$, so firms solve

$$\pi(p, w) = \sup_{z \in \mathbb{R}^m_+} \underbrace{pf(z)}_{\text{revenue}} - \underbrace{w \cdot z}_{\text{cost}}$$

$$z(p, w) = \operatorname*{argmax}_{z \in \mathbb{R}^m_+} pf(z) - w \cdot z$$

Dividing up the problem II

We separate the profit maximization problem into two parts:

1. Find a cost-minimizing way to produce a given output level q
   - Cost function
   $$c(q, w) \equiv \inf_{z\colon f(z) \geq q} w \cdot z$$
   - Conditional factor demand correspondence
   $$\begin{aligned} Z^*(q, w) &\equiv \operatorname*{argmin}_{z\colon f(z) \geq q} w \cdot z \\ &= \{z\colon f(z) \geq q \text{ and } w \cdot z = c(q, w)\} \end{aligned}$$
2. Find an output level that maximizes difference between revenue and cost
   $$\max_{q \geq 0} pq - c(q, w)$$

The Cost Minimization Problem I

Consider a restriction of Y that only includes output above some fixed level q:

$$Y_q \equiv \left\{(q, -z) \colon z \in \mathbb{R}^m_+ \text{ and } f(z) \geq q\right\}$$

The cost minimization problem is like a PMP over Y_q with

- $\pi_q(p, w) \equiv qp - c(q, w)$
- $y_q(w) \equiv \left[q \quad - Z^*(q, w)\right]$

The Cost Minimization Problem II

Our results from the profit maximization section go through here with appropriate sign changes; e.g.,

- $c(q, \cdot)$ is homogeneous of degree one in w
- $Z^*(q, \cdot)$ is homogeneous of degree zero in w
- If $Z^*(q, \cdot)$ is differentiable in w, then the matrix $D_w Z^*(q, w) = D_w^2 c(q, w)$ is symmetric and negative semidefinite
- Rationalizability condition. . .

The Cost Minimization Problem III

Theorem

Conditional factor demand function $z\colon \mathbb{R} \times W \rightrightarrows \mathbb{R}^n$ and differentiable cost function $c\colon \mathbb{R} \times W \to \mathbb{R}$ for a fixed output q on an open convex set $W \subseteq \mathbb{R}^m$ of input prices are jointly rationalizable iff

1. $c(q, w) = w \cdot z(q, w)$ *(adding-up);*
2. $\nabla_w c(q, w) = z(q, w)$ *(Shephard's Lemma);*
3. $c(q, \cdot)$ *is concave in* w *(for a fixed* q*).*

First-order conditions: PMP I

Single-output profit maximization problem

$$\max_{z \in \mathbb{R}^m_+} \underbrace{pf(z)}_{\text{revenue}} - \underbrace{w \cdot z}_{\text{cost}}$$

where $p > 0$ is the output price and $w \in \mathbb{R}^m_+$ are input prices.

Set up the Lagrangian and find Kuhn-Tucker conditions (assume differentiability):

$$\mathcal{L}(z, p, w, \mu) \equiv pf(z) - w \cdot z + \mu \cdot z$$

We get three (new) kinds of conditions. . .

First-order conditions: PMP II

1. FONCs: $p\nabla f(z^*) - w + \mu = \mathbf{0}$
2. Complementary slackness: $\mu_i z_i^* = 0$ for all i
3. Non-negativity: $\mu_i \geq 0$ for all i
4. Original constraints: $z_i^* \geq 0$ for all i

First three can be summarized as: for all i,

$$p\frac{\partial f(z^*)}{\partial z_i} \leq w_i \text{ with equality if } z_i^* > 0$$

First-order conditions: CMP

Single-output cost minimization problem

$$\min_{z \in \mathbb{R}^m_+} w \cdot z \text{ such that } f(z) \geq q.$$

$$\mathcal{L}(z, q, w, \lambda, \mu) \equiv -w \cdot z + \lambda\big(f(z) - q\big) + \mu \cdot z$$

Applying Kuhn-Tucker here gives

$$\lambda \frac{\partial f(z^*)}{\partial z_i} \leq w_i \text{ with equality if } z_i^* > 0$$

First-order conditions: Optimal Output Problem

Optimal output problem

$$\max_{q \geq 0} pq - c(q, w).$$

$$\mathcal{L}(q, p, w, \mu) \equiv pq - c(q, w) + \mu q$$

Applying Kuhn-Tucker here gives

$$p \leq \frac{\partial c(q^*, w)}{\partial q} \text{ with equality if } q^* > 0$$

Comparing the problems' Kuhn-Tucker conditions

- Profit Maximization Problem:

$$p\frac{\partial f(z^*)}{\partial z_i} \leq w_i \text{ with equality if } z_i^* > 0$$

- Cost Minimization Problem:

$$\lambda\frac{\partial f(z^*)}{\partial z_i} \leq w_i \text{ with equality if } z_i^* > 0$$

- Optimal Output Problem:

$$p \leq \frac{\partial c(q^*, w)}{\partial q} \text{ with equality if } q^* > 0$$

If $(q^*, z^*) > 0$, then p, λ, and $\frac{\partial c(q^*, w)}{\partial q}$ are all "the same"

Part III

Comparative Statics

Comparative statics

Comparative statics is the study of how endogenous variables respond to changes in exogenous variables

Endogenous variables are typically set by

1. Maximization, or
2. Equilibrium

Often we can characterize a maximization problem as a system of equations (like an equilibrium)

- Typically we do this using FOCs
- Key comparative statics tool is the Implicit Function Theorem
- Runs into lots of problems with continuity, smoothness, nonconvexity, et cetera

Since we often only care about directional statements, we will also cover monotone comparative statics tools

Comparative statics tools

We will discuss (and use throughout the quarter):

1. Envelope Theorem
2. Implicit Function Theorem
3. Topkis' Theorem
4. Monotone Selection Theorem

Outline

- Differentiable problems: the FOC approach
  - FOC-based comparative statics tools
  - Envelope Theorems
  - The Implicit Function Theorem
- Monotone comparative statics
  - Univariate
  - Multivariate
- Applications to producer theory

Outline

- Differentiable problems: the FOC approach
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Envelope Theorem

The ET and IFT tell us about the derivatives of different objects with respect to the parameters of the problem (i.e., exogenous variables):

- Envelope Theorems consider value function
- Implicit Function Theorem considers choice function

Envelope Theorem

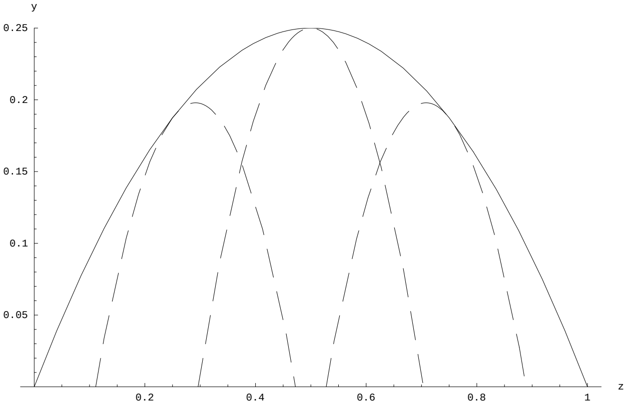
A simple Envelope Theorem:

$$
\begin{aligned}
v(q) &= \max_x f(x, q) \\
&= f\big(x_*(q), q\big) \\
\nabla_q v(q) &= \nabla_q f\big(x_*(q), q\big) + \underbrace{\nabla_x f\big(x_*(q), q\big)}_{=\mathbf{0}\text{ by FOC}} \cdot \nabla_q x_*(q) \\
&= \nabla_q f\big(x_*(q), q\big)
\end{aligned}
$$

Think of the ET as an application of the chain rule and then FOCs

Illustrating the Envelope Theorem

Objectives and envelope for $v(z) \equiv \max_x -5(x-z)^2 - z(z-1)$



A more complete Envelope Theorem

Theorem (Envelope Theorem)

Consider a constrained optimization problem $v(\theta) = \max_x f(x, \theta)$ *such that* $g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0$.

Comparative statics on the value function are given by:

$$\frac{\partial v}{\partial \theta_i} = \left.\frac{\partial f}{\partial \theta_i}\right|_{x^*} + \sum_{k=1}^{K} \lambda_k \left.\frac{\partial g_k}{\partial \theta_i}\right|_{x^*} = \left.\frac{\partial \mathcal{L}}{\partial \theta_i}\right|_{x^*}$$

(for Lagrangian $\mathcal{L}(x, \theta, \lambda) \equiv f(x, \theta) + \sum_k \lambda_k g_k(x, \theta)$*) for all* θ *such that the set of binding constraints does not change in an open neighborhood.*

Roughly, the derivative of the value function is the derivative of the Lagrangian

Example: Cost Minimization Problem

Single-output cost minimization problem

$$\min_{z \in \mathbb{R}^m_+} w \cdot z \text{ such that } f(z) \geq q.$$

$$\mathcal{L}(q, w, \lambda, \mu) \equiv -w \cdot z + \lambda\big(f(z) - q\big) + \mu \cdot z$$

Applying Kuhn-Tucker here gives

$$\lambda \frac{\partial f(z^*)}{\partial z_i} \leq w_i \text{ with equality if } z_i^* > 0$$

The ET applied to $c(q, w) \equiv \min_{z \in \mathbb{R}^m_+, f(z) \geq q} w \cdot z$ gives

$$\frac{\partial c(q, w)}{\partial q} = \lambda$$

The Implicit Function Theorem I

A simple, general maximization problem

$$X^*(t) = \operatorname*{argmax}_{x \in X} F(x, t)$$

where $F \colon X \times T \to \mathbb{R}$ and $X \times T \subseteq \mathbb{R}^2$.

Suppose:

1. Smoothness: F is twice continuously differentiable
2. Convex choice set: X is convex
3. Strictly concave objective (in choice variable): $F''_{xx} < 0$ (together with convexity of X, this ensures a unique maximizer)
4. Interiority: $x(t)$ is in the interior of X for all t (which means the standard FOC must hold)

The Implicit Function Theorem II

The first-order condition says the unique maximizer satisfies

$$F'_x\big(x(t), t\big) = 0$$

Taking the derivative in t:

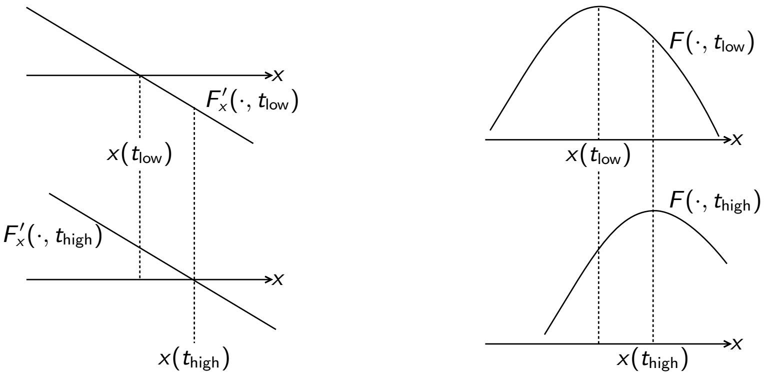
$$x'(t) = -\frac{F''_{xt}\big(x(t), t\big)}{F''_{xx}\big(x(t), t\big)}$$

Note by strict concavity, the denominator is negative, so $x'(t)$ and the cross-partial $F''_{xt}\big(x(t), t\big)$ have the same sign

Illustrating the Implicit Function Theorem

FOC: $F'_x\big(x(t), t\big) = 0$

Suppose $F''_{xt} > 0$ Thus $t_{\text{high}} > t_{\text{low}} \implies F'_x(x, t_{\text{high}}) > F'_x(x, t_{\text{low}})$



Intuition for the Implicit Function Theorem

When $F''_{xt} \geq 0$, an increase in x is more valuable when the parameter t is higher

In a sense, x and t are complements; we therefore expect that an increase in t results in an increase in the optimal choice of x

This intuition should carry through without all our assumptions

- MCS will lead us to the same conclusion without smoothness of F or strict concavity of F in x
- The sign of $x'(t)$ should be *ordinal* (i.e., invariant to monotone transformations of F)

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Motivating "increasing differences"

Recall the implicit function theorem relies on the cross-partial derivative of the objective function between the choice variable and the parameter

Key idea behind the implicit function theorem

"An increase in the choice variable is more valuable when the parameter is higher."

Consider

- Parameter values: $t < t'$
- Choice values: $x < x'$

The "value" of an increase in the choice variable is

- $F(x', t') - F(x, t')$ when the parameter is high (t')
- $F(x', t) - F(x, t)$ when the parameter is low (t)

Increasing differences I

Definition (increasing differences)

$F\colon X \times T \to \mathbb{R}$ (with $X,\ T \subseteq \mathbb{R}$) has (weakly) increasing differences iff for all $x' > x$ and $t' > t$

$$F(x', t') - F(x, t') \geq F(x', t) - F(x, t).$$

F has strictly/strongly increasing differences iff the above inequality is strict.

Note the definition is symmetric between x and t

Increasing differences II

Assuming $F(\cdot, \cdot)$ is sufficiently smooth, the following are equivalent:

1. F has increasing differences
2. $F'_x(x, t)$ is nondecreasing in t for all x
3. $F'_t(x, t)$ is nondecreasing in x for all t
4. $F''_{xt}(x, t) \geq 0$ for all (x, t)

Intuitively, ID means the variables enter the objective function in a "complementary" manner

Increasing differences III

If for all $x' > x$ and $t' > t$

$$F(x', t') - F(x, t') \leq F(x', t) - F(x, t),$$

we can say any of

- F has increasing differences in $(x, -t)$,
- F has increasing differences in $(-x, t)$, or
- $-F$ has increasing differences [in (x, t)]

Towards Topkis' Theorem I

Topkis will basically tell us that if the objective function has ID, the maximizer $x^*(t)$ will be increasing in parameter t

- If the maximizer is unique, this is exactly what Topkis says
- A slight wrinkle arises if the argmax is not always single-valued

Towards Topkis' Theorem II

Definition (strong set order)

$A \leq B$ in the strong set order iff for any $a \in A$ and $b \in B$,

$$a \geq b \implies b \in A \text{ and } a \in B,$$

or equivalently

$$\min\{a, b\} \in A \text{ and } \max\{a, b\} \in B.$$

That is, ranking the elements of $A \cup B$ from lowest to highest gives:

$$\underbrace{\bullet\bullet\quad\bullet\qquad\bullet}_{A\setminus B}\underbrace{\bullet\qquad\bullet\bullet\quad\bullet}_{A\cap B}\qquad\underbrace{\bullet\quad\bullet\bullet\quad\bullet}_{B\setminus A}.$$

Topkis' Theorem I

Theorem (Topkis' Theorem)

Suppose

1. $F\colon X \times T \to \mathbb{R}$ *(with* $X,\ T \subseteq \mathbb{R}$*) has ID,*
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then

$$\min\{x, x'\} \in X^*(t) \text{ and } \max\{x, x'\} \in X^*(t').$$

In other words, $X^*(t) \leq X^*(t')$ *in strong set order.*

This implies $\sup X^*(\cdot)$ and $\inf X^*(\cdot)$ are nondecreasing

If $X^*(\cdot)$ is single-valued, then $X^*(\cdot)$ is nondecreasing

Topkis' Theorem II

Proof.

If $x \leq x'$ the statement is trivial, so suppose $x > x'$; thus $\min\{x, x'\} = x'$ and $\max\{x, x'\} = x$.

$$\begin{aligned} F(x,t) &\geq F(x',t) \text{ because } x \in X^*(t), \text{ and} \\ F(x',t') &\geq F(x,t') \text{ because } x' \in X^*(t'). \end{aligned}$$

Adding these two gives

$$F(x,t) + F(x',t') \geq F(x',t) + F(x,t'),$$

while if $x > x'$, ID gives that (recall $t' > t$ and $x' < x$)

$$F(x,t) + F(x',t') \leq F(x',t) + F(x,t').$$

Thus all the above inequalities hold with equality, implying that $x \in X^*(t')$ and $x' \in X^*(t)$. □

Monotone Selection Theorem

The Monotone Selection Theorem is the analogue of Topkis' Theorem for *strictly* increasing differences

It says any selection from $X^*(t)$ is nondecreasing in t

Theorem (Monotone Selection Theorem)

Suppose

1. $F: X \times T \to \mathbb{R}$ *(with* $X,\ T \subseteq \mathbb{R}$*) has SID,*
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then $x' \geq x$.

Multidimensional increasing differences

Suppose we have more than one choice variable, e.g.,

$$\max_{(x_1,x_2)\in X\subseteq\mathbb{R}^2} F(x_1, x_2, t)$$

Topkis says

- F has ID in $(x_1, t) \Rightarrow x_1^*$ *holding* x_2 *fixed* is nondecreasing in t
- F has ID in $(x_2, t) \Rightarrow x_2^*$ *holding* x_1 *fixed* is nondecreasing in t
- *Nothing* if both x_1^* and x_2^* can respond to changes in t

There are indirect effects between the choice variables; they may reinforce or counteract the direct effects

- To apply multivariate Topkis, we also need effects to reinforce
- This also requires ID between x_1 and x_2

Lattice theory 101: Meet and Join

For x, $y \in \mathbb{R}^n$,

Definition (meet)

$$x \wedge y \equiv \big(\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}\big).$$

Definition (join)

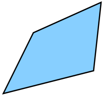
$$x \vee y \equiv \big(\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\}\big).$$

More generally, on a partially ordered set, $x \wedge y$ is the greatest lower bound of x and y, and $x \vee y$ is the least upper bound

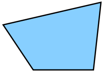
Lattice theory 101: Sublattices

Definition (sublattice)

Any set $X \subseteq \mathbb{R}^n$ such that for all x and $y \in X$, we have $x \wedge y \in X$ and $x \vee y \in X$.



Sublattice



Not

Lattice theory 101: Supermodularity

Definition (supermodularity)

$F \colon X \to \mathbb{R}^n$ on a sublattice X is supermodular iff for all $x, y \in X$, we have

$$F(x \wedge y) + F(x \vee y) \geq F(x) + F(y).$$

Supermodularity is equivalent to ID in all pairs of variables

Definition (submodularity)

$F \colon X \to \mathbb{R}^n$ on a sublattice X is submodular iff for all $x, y \in X$, we have

$$F(x \wedge y) + F(x \vee y) \leq F(x) + F(y).$$

Submodularity is equivalent to $-F$ having ID in all pairs of variables

Multivariate Topkis' Theorem I

Theorem (Topkis' Theorem)

Suppose

1. $F: X \times T \to \mathbb{R}$ *(for X a lattice, T fully ordered) is supermodular,*
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then

$$(x \wedge x') \in X^*(t) \text{ and } (x \vee x') \in X^*(t').$$

That is, $X^*(\cdot)$ is nondecreasing in t in the stronger set order.

Multivariate Topkis' Theorem II

Topkis' Theorem as stated on the last slide still makes unnecessary assumptions; in full generality it actually says

Theorem (Topkis' Theorem)

Suppose

1. $F: X \times T \to \mathbb{R}$ *(for X a lattice, T partially ordered)*
   - *is supermodular in x (i.e., ID in all (x_i, x_j))*
   - *has ID in (x, t) (i.e., ID in all (x_i, t_j))*
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then

$$(x \wedge x') \in X^*(t) \text{ and } (x \vee x') \in X^*(t').$$

That is, $X^*(\cdot)$ is nondecreasing in t in the stronger set order.

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Complement and substitute inputs I

Informally, two inputs are called

- Substitutes when an increase in the price of one leads to an increase in input demand for the other
- Complements when an increase in the price of one leads to a decrease in input demand for the other

If differentiable, given by sign of an element of substitution matrix:

- Substitutes: $\partial y_i / \partial p_j = \partial y_j / \partial p_i \leq 0$
- Complements: $\partial y_i / \partial p_j = \partial y_j / \partial p_i \geq 0$

(Recall sign convention: inputs are negative quantities)

Complement and substitute inputs II

Things are actually a bit more complicated. . .

1. May not be uniform (substitutes somewhere, complements elsewhere)
2. Which firm problem should we use?
   - All inputs and outputs can vary
   - Some inputs are held fixed ("short-run optimization")
   - Output is held fixed (cost-minimization problem)

Complement inputs I

Consider single-output profit-maximization with all inputs free to vary ("long-run optimization")

Theorem

Restrict attention to price vectors $(p, w) \in \mathbb{R}^n_+$ at which factor demand correspondence $z(p, w)$ is single-valued.

If production function $f(z)$ is increasing and supermodular, then $z(p, w)$ is

1. *Nondecreasing in p, and*
2. *Nonincreasing in w.*

Supermodularity of the production function implies price-theoretic complementarity of inputs

Complement inputs II

Proof.

Consider the firm's objective function $F(z, p, w) = pf(z) - w \cdot z$.

- $F'_p = f(z)$ is nondecreasing in z_i, hence F has ID in (z_i, p) for all i;
- $F''_{z_i\, w_i} = -1 \leq 0$, hence F has ID in $(z_i, -w_i)$ for all i; and
- $F''_{z_i\, w_j} = F''_{w_i\, w_j} = F''_{p\, w_i} = 0 \leq 0$, hence F has ID in $(z_i, -w_j)$, $(-w_i, -w_j)$, and $(p, -w_i)$ for all $i \neq j$.
- Supermodularity of F in z obtains from supermodularity of f, since $w \cdot z_1 + w \cdot z_2 = w \cdot (z_1 \vee z_2) + w \cdot (z_1 \wedge z_2)$ and $p \geq 0$.

Thus F is supermodular in $(z, p, -w)$; by Topkis' Theorem, $z(p, w)$ is nondecreasing in p and nonincreasing in w. □

Substitute inputs I

For the two-input case, we get an analogous result

Consider single-output profit-maximization with all inputs free to vary ("long-run optimization")

Theorem

Suppose there are only two inputs. Restrict attention to price vectors $(p, w) \in \mathbb{R}^n_+$ at which factor demand correspondence $z(p, w)$ is single-valued.

If production function $f(z)$ is increasing and submodular, then

1. $z_1(p, w)$ *is nondecreasing in* w_2, *and*
2. $z_2(p, w)$ *is nondecreasing in* w_1.

Submodularity of the production function implies price-theoretic substitutability of inputs in the two input case

Substitute inputs II

Proof.

$f(\cdot)$ is increasing and submodular, thus the firm's objective

$$pf(z_1, z_2) - w_1 z_1 - w_2 z_2$$

is supermodular in $(z_1, -z_2, w_2)$ and in $(z_2, -z_1, w_1)$.

By Topkis' Theorem, $z_1(p, w)$ is nondecreasing in w_2 and $z_2(p, w)$ is nondecreasing in w_1. □

If there are ≥ 3 inputs, feedback between inputs with unchanging prices makes for unpredictable results.

LeChâtelier principle

Samuelson's "LeChâtelier principle" claims that

- "Auxiliary constraints ('just-binding' in leaving initial equilibrium unchanged) reduce the response to a parameter change"
- That is, long-run reactions are larger than short-run reactions, since more inputs can be adjusted
  - In particular, firms react more to input price changes in the long-run than in the short-run

The principle does not consistently hold

LeChâtelier principle: Examples

Profit-maximization problem for a single-output firm with

- Two inputs: capital and labor
- Production function $f(k, l)$ with decreasing returns to scale (which implies $f(\cdot, \cdot)$ is concave)

Firm maximizes $pf(k, l) - rk - wl$ (PMP)...

- over l in short run (capital fixed)
- over l and k in long run

LeChâtelier principle: Example 1

Example

PMP for single-output firm with two inputs complements in the sense of supermodularity (i.e., $f''_{kl} \geq 0$ if f sufficiently smooth).

Suppose wages increase but capital is fixed in the short-run:

1. In the short run, labor goes down $(pf'_l(k^*_{\text{old}}, l^*_{\text{SR}}) = w_{\text{new}} > w_{\text{old}})$
2. In the long run, labor goes down and capital goes down by submodularity $(pf'_l(k^*_{\text{LR}}, l^*_{\text{LR}}) = w_{\text{new}} > w_{\text{old}})$
3. LR labor goes down more, since firm responds not only to higher wage, but also reduced capital stock with its resulting lower MPL

LeChâtelier principle: Example 2

Example

PMP for single-output firm with two inputs substitutes in the sense of submodularity (i.e., $f''_{kl} \leq 0$ if f sufficiently smooth).

Suppose wages increase but capital is fixed in the short-run:

1. In the short run, labor goes down $(pf'_l(k^*_{\text{old}}, l^*_{\text{SR}}) = w_{\text{new}} > w_{\text{old}})$
2. In the long run, labor goes down and capital goes up by submodularity $(pf'_l(k^*_{\text{LR}}, l^*_{\text{LR}}) = w_{\text{new}} > w_{\text{old}})$
3. LR labor goes down more, since firm responds not only to higher wage, but also higher capital stock with its resulting lower MPL

Does the LeChâtelier principle hold?

- Both examples above were characterized by a special property that made for a positive feedback loop
- The principle does not consistently hold; only if each pair of inputs are substitutes everywhere or complements everywhere

Theorem (LeChâtelier Principle)

Suppose twice differentiable production function $f(k, l)$ satisfies either $f''_{kl} \geq 0$ everywhere, or $f''_{kl} \leq 0$ everywhere. Then if wage w_l increases (decreases), the firm's labor demand will decrease (increase), and the decrease (increase) will be larger in the long-run than in the short-run.

- This can be described as a corollary of a more general principle proved in the lecture notes

Part IV

Choice Theory

Individual decision-making under certainty

Objects of inquiry

Our study begins with individual decision-making under certainty
Items of interest include:

- Feasible set
- Objective function (Feasible set $\to \mathbb{R}$)
- Choice correspondence (Parameters $\rightrightarrows$ Feasible set)
- "Maximized" objective function (Parameters $\to \mathbb{R}$)

We start with an even more general problem that only includes

- Feasible set
- Choice correspondence

A fairly innocent assumption will then allow us to treat this model as an optimization problem

Individual decision-making under certainty

Course outline

We will divide decision-making under certainty into three units:

1. Producer theory
   - Feasible set defined by technology
   - Objective function $p \cdot y$ depends on prices
2. Abstract choice theory
   - Feasible set totally general
   - Objective function may not even exist
3. Consumer theory
   - Feasible set defined by budget constraint and depends on prices
   - Objective function $u(x)$

Origins of rational choice theory

Choice theory aims to provide answers to

Positive questions Understanding how individual self-interest drives larger economic systems

Normative questions Objective criterion for utilitarian calculations

Values of the model

- Useful (somewhat)
  - Can often recover preferences from choices
  - Aligned with democratic values
  - But… interpersonal comparisons prove difficult
- Accurate (somewhat): many comparative statics results empirically verifiable
- Broad
  - Consumption and production
  - Lots of other things
- Compact
  - Extremely compact formulation
  - Ignores an array of other important "behavioral" factors

Simplifying assumptions

Very minimal:

1. Choices are made from some feasible set
2. Preferred things get chosen
3. Any pair of potential choices can be compared
4. Preferences are transitive
   (e.g., if apples are at least as good as bananas, and bananas are at least as good as cantaloupe, then apples are at least as good as cantaloupe)

Outline

- Preferences
  - Preference relations and rationality
  - From preferences to behavior
  - From behavior to preferences: "revealed preference"
- Utility functions
- Properties of preferences
- Behavioral critiques

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The set of all possible choices

We consider an entirely general set of possible choices

- Number of choices
  - Finite (e.g., types of drinks in my refrigerator)
  - Countably infinite (e.g., number of cars)
  - Uncountably infinite (e.g., amount of coffee)
  - Bounded *or* unbounded
- Order of choices
  - Fully ordered (e.g., years of schooling)
  - Partially ordered (e.g., AT&T cell phone plans)
  - Unordered (e.g., wives/husbands)

Note not all choices need be *feasible* in a particular situation

Preference relations

Definition (weak preference relation)

$\succsim$ is a binary relation on a set of possible choices X such that $x \succsim y$ iff "x is at least as good as y."

Definition (strict preference relation)

$\succ$ is a binary relation on X such that $x \succ y$ ("x is strictly preferred to y") iff $x \succsim y$ but $y \not\succsim x$.

Definition (indifference)

$\sim$ is a binary relation on X such that $x \sim y$ ("the agent is indifferent between x and y") iff $x \succsim y$ and $y \succsim x$.

Properties of preference relations

Definition (completeness)

$\succsim$ on X is complete iff $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$.

Completeness implies that $x \succsim x$

Definition (transitivity)

$\succsim$ on X is transitive iff whenever $x \succsim y$ and $y \succsim z$, we have $x \succsim z$.

Rules out preference cycles except in the case of indifference

Definition (rationality)

$\succsim$ on X is rational iff it is both complete and transitive.

Summary of preference notation

	$y \succsim x$	$y \not\succsim x$
$x \succsim y$	$x \sim y$	$x \succ y$
$x \not\succsim y$	$y \succ x$	Ruled out by completeness

Can think of (complete) preferences as inducing a function

$$p \colon X \times X \to \{\succ, \sim, \prec\}$$

Other properties of rational preference relations

Assume $\succsim$ is rational. Then for all x, y, $z \in X$:

- Weak preference is reflexive: $x \succsim x$
- Indifference is
  - Reflexive: $x \sim x$
  - Transitive: $(x \sim y) \wedge (y \sim z) \implies x \sim z$
  - Symmetric: $x \sim y \iff y \sim x$
- Strict preference is
  - Irreflexive: $x \not\succ x$
  - Transitive: $(x \succ y) \wedge (y \succ z) \implies x \succ z$
- $(x \succ y) \wedge (y \succsim z) \implies x \succ z$, and
  $(x \succsim y) \wedge (y \succ z) \implies x \succ z$

Two strategies for modelling individual decision-making

1. Conventional approach
   Start from preferences, ask what choices are compatible

2. Revealed-preference approach
   Start from observed choices, ask what preferences are compatible
   - Can we test rational choice theory? How?
   - Are choices consistent with maximization of some objective function? Can we recover an objective function?
   - How can we use objective function—in particular, do interpersonal comparisons work? If so, how?

Choice rules

Definition (Choice rule)

Given preferences $\succsim$ over X, and choice set $B \subseteq X$, the choice rule is a correspondence giving the set of all "best" elements in B:

$$C(B, \succsim) \equiv \{x \in B : x \succsim y \text{ for all } y \in B\}.$$

Theorem

Suppose $\succsim$ is complete and transitive and B finite and non-empty. Then $C(B, \succsim) \neq \varnothing$.

Proof of non-emptiness of choice correspondence

Proof.

Proof by mathematical induction on the number of elements in B.

Consider $|B| = 1$ so $B = \{x\}$; by completeness $x \succsim x$, so $x \in C(B, \succsim) \implies C(B, \succsim) \neq \varnothing$.

Suppose that for all $|B| = n \geq 1$, we have $C(B, \succsim) \neq \varnothing$. Consider A such that $|A| = n + 1$; thus $A = B \cup \{x\}$. We can consider some $y \in C(B, \succsim)$ by the inductive hypothesis. By completeness, either

1. $y \succsim x$, in which case $y \in C(A, \succsim)$.
2. $x \succsim y$, in which case $x \in C(A, \succsim)$ by transitivity.

Thus $C(A, \succsim) \neq \varnothing$

The inductive hypothesis holds for all finite n. □

Revealed preference

- Before, we used a known preference relation $\succsim$ to generate choice rule $C(\cdot, \succsim)$
- Now we suppose the agent reveals her preferences through her choices, which we observe; can we deduce a rational preference relation that could have generated them?

Definition (revealed preference choice rule)

Any $C_R \colon 2^X \to 2^X$ (where 2^X means the set of subsets of X) such that for all $A \subseteq X$, we have $C_R(A) \subseteq A$.

If $C_R(\cdot)$ could be generated by a rational preference relation (i.e., there exists some complete, transitive $\succsim$ such that $C_R(A) = C(A, \succsim)$ for all A), we say it is rationalizable

Examples of revealed preference choice rules

Suppose we know $C_R(\cdot)$ for

- $A \equiv \{a, b\}$
- $B \equiv \{a, b, c\}$

$C_R(\{a, b\})$	$C_R(\{a, b, c\})$	Possibly rationalizable?	
$\{a\}$	$\{c\}$	✓	$(c \succ a \succ b)$
$\{a\}$	$\{a\}$	✓	$(a \succ b,\ a \succ c,\ b ? c)$
$\{a, b\}$	$\{c\}$	✓	$(c \succ a \sim b)$
$\{c\}$	$\{c\}$	X	$(c \notin \{a, b\})$
$\varnothing$	$\{c\}$	X	(No possible $a?b$)
$\{b\}$	$\{a\}$	X	(No possible $a?b$)
$\{a\}$	$\{a, b\}$	X	(No possible $a?b$)

A necessary condition for rationalizability

Suppose that $C_R(\cdot)$ is rationalizable (in particular, it is generated by $\succsim$), and we observe $C_R(A)$ for some $A \subseteq X$ such that

- $a \in C_R(A)$ (a was chosen $\iff a \succsim z$ for all $z \in A$)
- $b \in A$ (b could have been chosen)

We can infer that $a \succsim b$

Now consider some $B \subseteq X$ such that

- $a \in B$
- $b \in C_R(B)$ (b was chosen $\iff b \succsim z$ for all $z \in B$)

We can infer that $b \succsim a$

Thus $a \sim b$, hence $a \in C_R(B)$ and $b \in C_R(A)$ by transitivity

Houthaker's Axiom of Revealed Preferences

A rationalizable choice rule $C_R(\cdot)$ must therefore satisfy "HARP":

Definition (Houthaker's Axiom of Revealed Preferences)

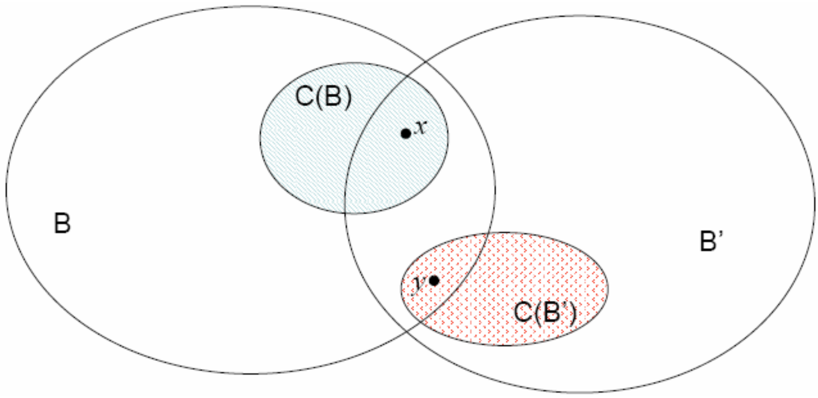
Revealed preferences $C_R \colon 2^X \to 2^X$ satisfies HARP iff $\forall a,\ b \in X$ and $\forall A,\ B \subseteq X$ such that

- $\{a, b\} \subseteq A$ and $a \in C_R(A)$; and
- $\{a, b\} \subseteq B$ and $b \in C_R(B)$,

we have that $a \in C_R(B)$ (and $b \in C_R(A)$).

Illustrating HARP

A violation of HARP:



Example of HARP

Suppose

1. Revealed preferences $C_R(\cdot)$ satisfy HARP, and that
2. $C_R(\cdot)$ is nonempty-valued (except for $C_R(\varnothing)$)

- If $C_R(\{a,b\}) = \{b\}$, what can we conclude about $C_R(\{a,b,c\})$?

$$C_R(\{a,b,c\}) \in \{\{b\},\{c\},\{b,c\}\}$$

- If $C_R(\{a,b,c\}) = \{b\}$, what can we conclude about $C_R(\{a,b\})$?

$$C_R(\{a,b\}) = \{b\}$$

HARP is necessary and sufficient for rationalizability I

Theorem

Suppose revealed preference choice rule $C_R \colon 2^X \to 2^X$ is nonempty-valued (except for $C_R(\varnothing)$). Then $C_R(\cdot)$ satisfies HARP iff there exists a rational preference relation $\succsim$ such that $C_R(\cdot) = C(\cdot, \succsim)$.

Proof.

Rationalizability $\Rightarrow$ HARP as argued above.
Rationalizability $\Leftarrow$ HARP: suppose $C_R(\cdot)$ satisfies HARP, we will construct a "revealed preference relation" $\succsim_c$ that generates $C_R(\cdot)$.
For any x and y, let $x \succsim_c y$ iff there exists some $A \subseteq X$ such that $y \in A$ and $x \in C_R(A)$.
We must show that $\succsim_c$ is complete, transitive, and generates C (i.e., $C_R(\cdot) = C(\cdot, \succsim_c)$).

HARP is necessary and sufficient for rationalizability II

Proof (continued).

1. $C_R(\cdot)$ is nonempty-valued, so either $x \in C_R(\{x, y\})$ or $y \in C_R(\{x, y\})$. Thus either $x \succsim_c y$ or $y \succsim_c x$.
2. Suppose $x \succsim_c y \succsim_c z$ and consider $C_R(\{x, y, z\}) \neq \varnothing$. Thus one (or more) of the following must hold:
    1. $x \in C_R(\{x, y, z\}) \implies x \succsim_c z$.
    2. $y \in C_R(\{x, y, z\})$.
       But $x \succsim_c y$, so by HARP $x \in C_R(\{x, y, z\}) \implies x \succsim_c z$ by 1.
    3. $z \in C_R(\{x, y, z\})$.
       But $y \succsim_c z$, so by HARP $y \in C_R(\{x, y, z\}) \implies x \succsim_c z$ by 2.
3. We must show that $x \in C_R(B)$ iff $x \succsim_c y$ for all $y \in B$.
    - $x \in C_R(B) \implies x \succsim_c y$ by construction of $\succsim_c$.
    - By nonempty-valuedness, there must be some $y \in C_R(B)$; by HARP, $x \succsim_c y$ implies that $x \in C_R(B)$. □

Revealed preference and limited data

Our discussion relies on all preferences being observed... real data is typically more limited

- *All* elements of $C_R(A)$... we may only see *one* element of A i.e., $\widetilde{C}_R(A) \in C_R(A)$
- $C_R(A)$ for *every* $A \subseteq X$... we may only observe choices for *certain* choice sets i.e., $\widehat{C}_R(A) \colon \mathcal{B} \to 2^X$ for $\mathcal{B} \subset 2^X$ with $\widehat{C}_R(A) = C_R(A)$

Other "axioms of revealed preference" hold in these environments

- Weak Axiom of Revealed Preference (WARP)
- Generalized Axiom of Revealed Preference (GARP)—necessary and sufficient condition for rationalizability

Weak Axiom of Revealed Preferences I

Definition (Weak Axiom of Revealed Preferences)

Revealed preferences $\widehat{C}_R : \mathcal{B} \to 2^X$ defined only for choice sets $\mathcal{B} \subseteq 2^X$ satisfies WARP iff $\forall a, b \in X$ and $\forall A, B \in \mathcal{B}$ such that

- $\{a, b\} \subseteq A$ and $a \in \widehat{C}_R(A)$; and
- $\{a, b\} \subseteq B$ and $b \in \widehat{C}_R(B)$,

we have that $a \in \widehat{C}_R(B)$ (and $b \in \widehat{C}_R(A)$).

HARP is WARP with all possible choice sets (i.e,. $\mathcal{B} = 2^X$)

WARP is necessary but not sufficient for rationalizability

Weak Axiom of Revealed Preferences II

WARP is not sufficient for rationalizability

Example

Consider $\widehat{C}_R \colon \mathcal{B} \to 2^{\{a,b,c\}}$ defined for choice sets $\mathcal{B} \equiv \big\{\{a,b\},\{b,c\},\{c,a\}\big\} \subseteq 2^{\{a,b,c\}}$ with:

- $\widehat{C}_R\big(\{a,b\}\big) = \{a\}$,
- $\widehat{C}_R\big(\{b,c\}\big) = \{b\}$, and
- $\widehat{C}_R\big(\{c,a\}\big) = \{c\}$.

$\widehat{C}_R(\cdot)$ satisfies WARP, but is not rationalizable.

Think of $\widehat{C}_R(\cdot)$ as a restriction of some $C_R \colon 2^{\{a,b,c\}} \to 2^{\{a,b,c\}}$; there is no $C_R\big(\{a,b,c\}\big)$ consistent with HARP

Outline

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From abstract preferences to maximization

- Our model of choice so far is entirely abstract
- Utility assigns a numerical ranking to each possible choice
- By assigning a utility to each element of X, we turn the choice problem into an optimization problem

Definition (utility function)

Utility function $u\colon X \to \mathbb{R}$ represents $\succsim$ on X iff for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

Then the choice rule is

$$C(B, \succsim) \equiv \{x \in B\colon x \succsim y \text{ for all } y \in B\} = \operatorname*{argmax}_{x \in B} u(x)$$

Utility representation implies rationality

Theorem

If utility function $u\colon X \to \mathbb{R}$ represents $\succsim$ on X, then $\succsim$ is rational.

Proof.

For any $x, y \in X$, we have $u(x), u(y) \in \mathbb{R}$, so either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. Since $u(\cdot)$ represents $\succsim$, either $x \succsim y$ or $y \succsim x$; i.e., $\succsim$ is complete.

Suppose $x \succsim y \succsim z$. Since $u(\cdot)$ represents $\succsim$, we know $u(x) \geq u(y) \geq u(z)$. Thus $u(x) \geq u(z) \implies x \succsim z$. $\succsim$ is transitive. $\square$

Ordinality of utility and interpersonal comparisons

Note that $\succsim$ is represented by any function satisfying

$$x \succsim y \iff u(x) \geq u(y)$$

for all $x, y \in X$

Thus any increasing monotone transformation of $u(\cdot)$ also represents $\succsim$

- The property of representing $\succsim$ is ordinal
- There is no such thing as a "util"

Failure of interpersonal comparisons

Interpersonal comparisons are impossible using this theory

1. Disappointing to original utilitarian agenda
2. Rawls (following Kant, following. . .) attempts to solve this by asking us to consider only a single chooser
3. "Just noticeable difference" suggests defining one util as the smallest difference an individual can notice
   - $x \succsim y$ iff $u(x) \geq u(y) - 1$
   - Note $\succ$ is transitive, but $\succsim$ is not

Can we find a utility function representing $\succsim$? I

Theorem

Any complete and transitive preference relation $\succsim$ on a finite set X can be represented by some utility function $u\colon X \to \{1, \ldots, n\}$ where $n \equiv |X|$.

Intuitive argument:

1. Assign the "top" elements of X utility $n = |X|$
2. Discard them; we are left with a set X'
3. If $X' = \varnothing$, we are done; otherwise return to step 1 with the set X'

Can we find a utility function representing $\succsim$? II

Proof.

Proof by mathematical induction on $n \equiv |X|$. The theorem holds trivially for $n = 0$, since $X = \varnothing$.

Suppose the theorem holds for sets with at most n elements. Consider a set X with $n + 1$ elements. $C(X, \succsim) \neq \varnothing$, so $Y \equiv X \setminus C(X, \succsim)$ has at most n elements. By inductive hypothesis, preferences on Y are represented by some $u \colon Y \to \{1, \dots, n\}$. We extend u to X by setting $u(x) = n + 1$ for all $x \in C(X, \succsim)$.

We must show that this extended u represents $\succsim$ on X.

Can we find a utility function representing $\succsim$? III

Proof (continued).

We must show that this extended u represents $\succsim$ on X; i.e., that for all x and $y \in X$, we have $x \succsim y$ iff $u(x) \geq u(y)$.

- $x \succsim y \implies u(x) \geq u(y)$. If $x \in C(X, \succsim)$, then $u(x) = n + 1 \geq u(y)$. If $x \notin C(X, \succsim)$, then by transitivity $y \notin C(X, \succsim)$, so x and $y \in Y$. Since u represents $\succsim$ on Y, we must have $u(x) \geq u(y)$.
- $x \succsim y \impliedby u(x) \geq u(y)$. If $u(x) = n + 1$, then $x \in C(X, \succsim)$, hence $x \succsim y$. If $u(x) \leq n$, then x and $y \in Y$. Since u represents $\succsim$ on Y, we must have $x \succsim y$.

Thus the inductive hypothesis holds for all finite n. □

What if $|X| = \infty$? I

If X is infinite, our proof doesn't go through, but we still *may* be able to represent $\succsim$ by a utility function

Example

Preferences over $\mathbb{R}_+$ with $x_1 \succsim x_2$ iff $x_1 \geq x_2$.

$\succsim$ can be represented by $u(x) = x$. (It can also be represented by other utility functions.)

What if $|X| = \infty$? II

However, if X is infinite we can't necessarily represent $\succsim$ by a utility function

Example (lexicographic preferences)

Preferences over $[0,1]^2 \subseteq \mathbb{R}^2$ with $(x_1, y_1) \succsim (x_2, y_2)$ iff

- $x_1 > x_2$, or
- $x_1 = x_2$ and $y_1 \geq y_2$.

Lexicographic preferences can't be represented by a utility function

- There are no indifference curves
- A utility function would have to be an order-preserving one-to-one mapping from the unit square to the real line (both are infinite, but they are "different infinities")

Continuous preferences I

Definition (continuous preference relation)

A preference relation $\succsim$ on X is continuous iff for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $x_n \succsim y_n$ for all n,

$$\lim_{n\to\infty} x_n \succsim \lim_{n\to\infty} y_n.$$

Equivalently, $\succsim$ is continuous iff for all $x \in X$, the upper and lower contour sets of x

$$\text{UCS}(x) \equiv \{\xi \in X : \xi \succsim x\}$$
$$\text{LCS}(x) \equiv \{\xi \in X : x \succsim \xi\}$$

are both closed sets.

Continuous preferences II

Theorem

A continuous, rational preference relation $\succsim$ on $X \subseteq \mathbb{R}^n$ can be represented by a continuous utility function $u \colon X \to \mathbb{R}$.

(Note it may also be represented by noncontinuous utility functions)

Full proof in Debron and MWG; abbreviated proof in notes

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Reasons for restricting preferences

Analytical tractability often demands restricting "allowable" preferences

- Some restrictions are mathematical conveniences and cannot be empirically falsified (e.g., continuity)
- Some hold broadly (e.g., monotonicity)
- Some require situational justification

Restrictions on preferences imply restrictions on utility functions

Assumptions for the rest of this section

1. $\succsim$ is rational (i.e., complete and transitive).
2. For simplicity, we assume preferences over $X \subseteq \mathbb{R}^n$.

Properties of rational $\succsim$: Nonsatiation

Definition (monotonicity)

$\succsim$ is monotone iff $x > y \implies x \succsim y$. (N.B. MWG differs)
$\succsim$ is strictly monotone iff $x > y \implies x \succ y$.

i.e., more of something is (strictly) better

Definition (local non-satiation)

$\succsim$ is locally non-satiated iff for any y and $\varepsilon > 0$, there exists x such that $\|x - y\| \leq \varepsilon$ and $x \succ y$.

implies there are no "thick" indifference curves
$\succsim$ is locally non-satiated iff $u(\cdot)$ has no local maxima in X

Properties of rational $\succsim$: Convexity I

Convex preferences capture the idea that agents like diversity

1. Satisfying in some ways: rather alternate between juice and soda than have either one every day

2. Unsatisfying in others: rather have a glass of either one than a mixture

3. Key question is granularity of goods aggregation
   - Over time? What period?
   - Over what "bite size"?

Properties of rational $\succsim$: Convexity II

Definition (convexity)

$\succsim$ is convex iff $x \succsim y$ and $x' \succsim y$ together imply that

$$tx + (1-t)x' \succsim y \text{ for all } t \in (0,1).$$

Equivalently, $\succsim$ is convex iff the upper contour set of any y (i.e., $\{x \in X : x \succsim y\}$) is a convex set.

$\succsim$ is strictly convex iff $x \succsim y$ and $x' \succsim y$ (with $x \neq x'$) together imply that

$$tx + (1-t)x' \succ y \text{ for all } t \in (0,1).$$

i.e., one never gets worse off by mixing goods

$\succsim$ is (strictly) convex iff $u(\cdot)$ is (strictly) quasiconcave

Implications for utility representation

Property of $\succsim$	Property of $u(\cdot)$
Monotone	Nondecreasing
Strictly monotone	Increasing
Locally non-satiated	Has no local maxima in X
Convex	Quasiconcave
Strictly convex	Strictly quasiconcave

Properties of rational $\succsim$: Homotheticity

Definition (homotheticity)

$\succsim$ is homothetic iff for all x, y, and all $\lambda > 0$,

$$x \succsim y \iff \lambda x \succsim \lambda y.$$

Continuous, strictly monotone $\succsim$ is homothetic iff it can be represented by a utility function that is homogeneous of degree one (note it can also be represented by utility functions that aren't)

Properties of rational $\succsim$: Separability I

Suppose rational $\succsim$ over $X \times Y \subseteq \mathbb{R}^{p+q}$

- First p goods form some "group" $x \in X \subseteq \mathbb{R}^p$
- Other goods $y \in Y \subseteq \mathbb{R}^q$

Separable preferences

"Preferences over X do not depend on y" means that

$$(x', y_1) \succsim (x, y_1) \iff (x', y_2) \succsim (x, y_2)$$

for all x, $x' \in X$ and all y_1, $y_2 \in Y$.
Note the definition is *not* symmetric in X and Y.

The critical assumption for empirical analysis of preferences

Properties of rational $\succsim$: Separability II

Example

$X = \{\text{wine}, \text{beer}\}$ and $Y = \{\text{cheese}, \text{pretzels}\}$ with strict preference ranking

1. $(\text{wine}, \text{cheese}) \succ$
2. $(\text{wine}, \text{pretzels}) \succ$
3. $(\text{beer}, \text{pretzels}) \succ$
4. $(\text{beer}, \text{cheese})$.

Utility representation of separable preferences: theorem

Theorem

Suppose $\succsim$ on $X \times Y$ is represented by $u(x, y)$. Then preferences over X do not depend on y iff there exist functions $v : X \to \mathbb{R}$ and $U : \mathbb{R} \times Y \to \mathbb{R}$ such that

1. *$U(\cdot, \cdot)$ is increasing in its first argument, and*
2. *$u(x, y) = U\big(v(x), y\big)$ for all (x, y).*

Utility representation of separable preferences: example

Example

Preferences over beverages do not depend on your snack, and are represented by $u(\cdot,\cdot)$, where

$$\begin{aligned} u(\text{wine}, \text{cheese}) &= 4 \\ u(\text{wine}, \text{pretzels}) &= 3 \\ u(\text{beer}, \text{pretzels}) &= 2 \\ u(\text{beer}, \text{cheese}) &= 1. \end{aligned} \qquad \text{And let} \qquad \begin{aligned} U(3, \text{cheese}) &\equiv 4 \\ U(3, \text{pretzels}) &\equiv 3 \\ U(2, \text{pretzels}) &\equiv 2 \\ U(2, \text{cheese}) &\equiv 1. \end{aligned}$$

Let $v(\text{wine}) \equiv 3$ and $v(\text{beer}) \equiv 2$.

Thus

1. $U(\cdot,\cdot)$ is increasing in its first argument, and
2. $u(x, y) = U\big(v(x), y\big)$ for all (x, y).

Utility representation of separable preferences: proof I

Proof.

Conditions $\implies$ separability: If $u(x, y) = U\big(v(x), y\big)$ with $U(\cdot, \cdot)$ increasing in its first argument, then preferences over X given any y are represented by $v(x)$ and do not depend on y.

Conditions $\impliedby$ separability: We assume preferences over X do not depend on y, construct a U and v, and then show that they satisfy

1. $u(x, y) = U\big(v(x), y\big)$ for all (x, y), and
2. $U(\cdot, \cdot)$ is increasing in its first argument.

Utility representation of separable preferences: proof II

Proof (continued).

- Fix some $y_0 \in Y$, and let $v(x) \equiv u(x, y_0)$.
- Consider every α in the range of $v(\cdot)$; that is there is (at least one) $v^{-1}(\alpha)$ such that $v\big(v^{-1}(\alpha)\big) = \alpha$. Define

$$U(\alpha, y) \equiv u\big(v^{-1}(\alpha), y\big). \tag{1}$$

Note that

$$\begin{aligned} u\big(v^{-1}\big(v(x)\big), y_0\big) = v\big(v^{-1}\big(v(x)\big)\big) = v(x) &= u(x, y_0) \\ \big(v^{-1}\big(v(x)\big), y_0\big) \sim &\quad (x, y_0) \\ \big(v^{-1}\big(v(x)\big), y\big) \sim &\quad (x, y). \end{aligned} \tag{2}$$

Utility representation of separable preferences: proof III

Proof (continued).

By 1 and 2,

$$U(v(x), y) = u(v^{-1}(v(x)), y) = u(x, y).$$

Choose any $y \in Y$ and any x, $x' \in X$ such that $v(x') > v(x)$:

$$\begin{aligned} u(x', y_0) > u(x, y_0) \Longrightarrow (x', y_0) &\succ (x, y_0) \\ (x', y) &\succ (x, y) \\ u(x', y) &> u(x, y) \\ U(v(x'), y) &> U(v(x), y) \end{aligned}$$

so $U(\cdot, \cdot)$ is increasing in its first argument for all y. □

Properties of rational $\succsim$: Quasi-linearity I

Suppose rational $\succsim$ over $X \equiv \mathbb{R} \times Y$

- First good is the numeraire (a.k.a. "good zero" or "good one," confusingly): think money
- Other goods general; need not be in $\mathbb{R}^n$

Theorem

Suppose rational $\succsim$ on $X \equiv \mathbb{R} \times Y$ satisfies the "numeraire properties":

1. *Good 1 is valuable: $(t, y) \succsim (t', y) \iff t \geq t'$ for all y;*
2. *Compensation is possible: For every y, $y' \in Y$, there exists some $t \in \mathbb{R}$ such that $(0, y) \sim (t, y')$;*
3. *No wealth effects: If $(t, y) \succsim (t', y')$, then for all $d \in \mathbb{R}$, $(t + d, y) \succsim (t' + d, y')$.*

...

Properties of rational $\succsim$: Quasi-linearity II

Theorem (continued.)

Then there exists a utility function representing $\succsim$ of the form $u(t, y) = t + v(y)$ for some $v\colon Y \to \mathbb{R}$. (Note it can also be represented by utility functions that aren't of this form.)

Conversely, any $\succsim$ on $X = \mathbb{R} \times Y$ represented by a utility function of the form $u(t, y) = t + v(y)$ satisfies the above properties.

Properties of rational $\succsim$: Quasi-linearity III

Proof.

Suppose the numeraire properties hold. Fix some $\bar{y} \in Y$. Define a function $v\colon Y \to \mathbb{R}$ such that $(0, y) \sim \big(v(y), \bar{y}\big)$; this is possible by condition 2.

By condition 3, for any (t, y) and (t', y'), we have $(t, y) \sim \big(t + v(y), \bar{y}\big)$ and $(t', y') \sim \big(t' + v(y'), \bar{y}\big)$. Thus $(t, y) \succsim (t', y')$ iff $\big(t + v(y), \bar{y}\big) \succsim \big(t' + v(y'), \bar{y}\big)$ (by transitivity), which holds by condition 1 iff $t + v(y) \geq t' + v(y')$.

The converse is trivial. □

Outline

- Preferences
  - Preference relations and rationality
  - From preferences to behavior
  - From behavior to preferences: “revealed preference”
- Utility functions
- Properties of preferences
- Behavioral critiques

Problems with rational choice

Rational choice theory plays a central role in most tools of economic analysis

But. . . significant research calls into question underlying assumptions, identifying and explaining deviations using

- Psychology
- Sociology
- Cognitive neuroscience ("neuroeconomics")

Context-dependent choice

Choices appear to be highly situational, depending on

1. Other available options
2. Way that options are "framed"
3. Social context/emotional state

Numerous research projects consider these effects in real-world and laboratory settings

Non-considered choice

Rational choice theory depends on a *considered* comparison of options

- Pairwise comparison
- Utility maximization

Many actual choices appear to be made using

1. Intuitive reasoning
2. Heuristics
3. Instinctive desire

Part V

Consumer Theory 1

Individual decision-making under certainty

Course outline

We will divide decision-making under certainty into three units:

1. Producer theory
   - Feasible set defined by technology
   - Objective function $p \cdot y$ depends on prices
2. Abstract choice theory
   - Feasible set totally general
   - Objective function may not even exist
3. Consumer theory
   - Feasible set defined by budget constraint and depends on prices
   - Objective function $u(x)$

The consumer problem

Utility Maximization Problem

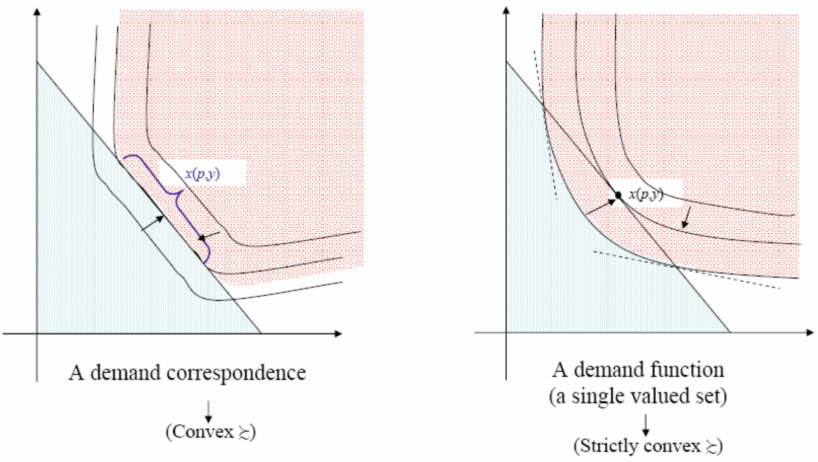
$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ such that } \underbrace{p \cdot x}_{\text{Expenses}} \leq w$$

where p are the prices of goods and w is the consumer's "wealth."

This type of choice set is a budget set

$$B(p, w) \equiv \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$$

Illustrating the Utility Maximization Problem



Assumptions underlying the UMP

Note that

- Utility function is general (but assumed to exist—a restriction of preferences)
- Choice set defined by linear budget constraint
  - Consumers are price takers
  - Prices are linear
  - Perfect information: prices are all known
- Finite number of goods
  - Goods are described by quantity and price
  - Goods are divisible
  - Goods may be time- or situation-dependent
  - Perfect information: goods are all well understood

Outline

- The utility maximization problem
  - Marshallian demand and indirect utility
  - First-order conditions of the UMP
  - Recovering demand from indirect utility
- The expenditure minimization problem
- Wealth and substitution effects
  - The Slutsky equation
  - Comparative statics properties

Outline

- The utility maximization problem
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Utility maximization problem

The consumer's Marshallian demand is given by correspondence $x\colon \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n_+$

$$\begin{aligned} x(p,w) &\equiv \operatorname*{argmax}_{x \in \mathbb{R}^n_+ \colon p \cdot x \leq w} u(x) \equiv \operatorname*{argmax}_{x \in B(p,w)} u(x) \\ &= \{x \in \mathbb{R}^n_+ \colon p \cdot x \leq w \text{ and } u(x) = v(p,w)\} \end{aligned}$$

Resulting indirect utility function is given by

$$v(p,w) \equiv \sup_{x \in \mathbb{R}^n_+ \colon p \cdot x \leq w} u(x) \equiv \sup_{x \in B(p,w)} u(x)$$

Properties of Marshallian demand and indirect utility

Theorem

$v(p, w)$ and $x(p, w)$ are homogeneous of degree zero. That is, for all p, w, and $\lambda > 0$,

$$v(\lambda p, \lambda w) = v(p, w) \text{ and } x(\lambda p, \lambda w) = x(p, w).$$

These are "no money illusion" conditions

Proof.

$B(\lambda p, \lambda w) = B(p, w)$, so consumers are solving the same problem. □

Implications of restrictions on preferences: continuity

Theorem

If preferences are continuous, $x(p, w) \neq \varnothing$ for every $p \gg \mathbf{0}$ and $w \geq 0$.

i.e., Consumers choose *something*

Proof.

$B(p, w) \equiv \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ is a closed, bounded set. Continuous preferences can be represented by a continuous utility function $\tilde{u}(\cdot)$, and a continuous function achieves a maximum somewhere on a closed, bounded set. Since $\tilde{u}(\cdot)$ represents the same preferences as $u(\cdot)$, we know $\tilde{u}(\cdot)$ must achieve a maximum precisely where $u(\cdot)$ does. $\square$

Implications of restrictions on preferences: convexity I

Theorem

If preferences are convex, then $x(p, w)$ is a convex set for every $p \gg \mathbf{0}$ and $w \geq 0$.

Proof.

$B(p, w) \equiv \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ is a convex set.
If x, $x' \in x(p, w)$, then $x \sim x'$.
For all $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)x' \in B(p, w)$ by convexity of $B(p, w)$ and $\lambda x + (1 - \lambda)x' \succsim x$ by convexity of preferences. Thus

$$\lambda x + (1 - \lambda)x' \in x(p, w).$$

□

Implications of restrictions on preferences: convexity II

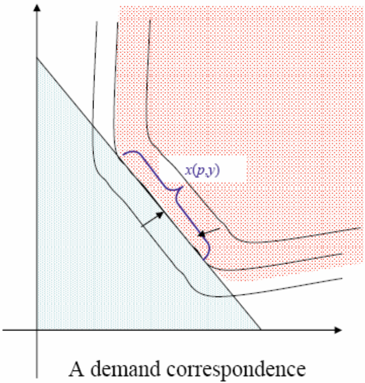
Theorem

If preferences are strictly convex, then $x(p, w)$ is single-valued for every $p \gg \mathbf{0}$ and $w \geq 0$.

Proof.

$B(p, w) \equiv \{x \in \mathbb{R}^n_+ : p \cdot x \leq w\}$ is a convex set.
If x, $x' \in x(p, w)$, then $x \sim x'$. Suppose $x \neq x'$.
For all $\lambda \in (0, 1)$, we have $\lambda x + (1 - \lambda)x' \in B(p, w)$ by convexity of $B(p, w)$ and $\lambda x + (1 - \lambda)x' \succ x$ by convexity of preferences.
But this contradicts the fact that $x \in x(p, w)$. Thus $x = x'$. □

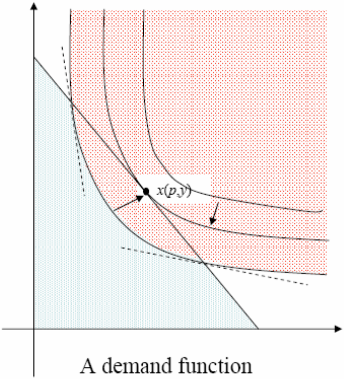
Implications of restrictions on preferences: convexity III



A demand correspondence

↓

(Convex ≿)



A demand function
(a single valued set)

↓

(Strictly convex ≿)

Implications of restrictions on preferences: non-satiation I

Definition (Walras' Law)

$p \cdot x = w$ for every $p \gg \mathbf{0}$, $w \geq 0$, and $x \in x(p, w)$.

Theorem

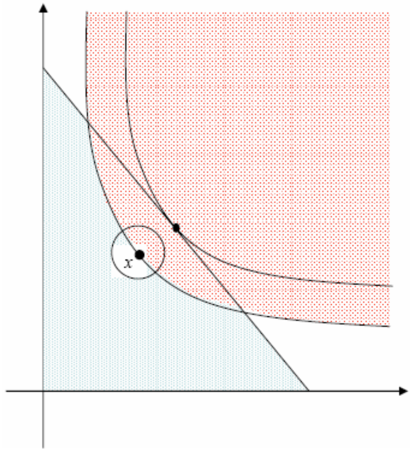
If preferences are locally non-satiated, then Walras' Law holds.

This allows us to replace the inequality constraint in the UMP with an equality constraint

Implications of restrictions on preferences: non-satiation II

Proof.

Suppose that $p \cdot x < w$ for some $x \in x(p, w)$. Then there exists some x' sufficiently close to x with $x' \succ x$ and $p \cdot x' < w$, which contradicts the fact that $x \in x(p, w)$. Thus $p \cdot x = w$. $\square$



Solving for Marshallian demand I

Suppose the utility function is differentiable

- This is an ungrounded assumption
- However, differentiability can not be falsified by any finite data set
- Also, utility functions are robust to monotone transformations

We may be able to use Kuhn-Tucker to "solve" the UMP:

Utility Maximization Problem

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ such that } p \cdot x \leq w$$

gives the Lagrangian

$$\mathcal{L}(x, \lambda, \mu, p, w) \equiv u(x) + \lambda(w - p \cdot x) + \mu \cdot x.$$

Solving for Marshallian demand II

1. First order conditions:

$$u_i'(x^*) = \lambda p_i - \mu_i \text{ for all } i$$

2. Complementary slackness:

$$\lambda(w - p \cdot x^*) = 0$$
$$\mu_i x_i^* = 0 \text{ for all } i$$

3. Non-negativity:

$$\lambda \geq 0 \text{ and } \mu_i \geq 0 \text{ for all } i$$

4. Original constraints $p \cdot x^* \leq w$ and $x_i^* \geq 0$ for all i

We can solve this system of equations for certain functional forms of $u(\cdot)$

The power (and limitations) of Kuhn-Tucker

Kuhn-Tucker provides conditions on (x, λ, μ) given (p, w):

1. First order conditions
2. Complementary slackness
3. Non-negativity
4. (Original constraints)

Kuhn-Tucker tells us that if x^* is a solution to the UMP, there exist some (λ, μ) such that these conditions hold; however:

- These are only necessary conditions; there may be (x, λ, μ) that satisfy Kuhn-Tucker conditions but do not solve UMP
- If $u(\cdot)$ is concave, conditions are necessary and sufficient

When are Kuhn-Tucker conditions sufficient?

Kuhn-Tucker conditions are necessary and sufficient for a solution (assuming differentiability) as long as we have a "convex problem":

1. The constraint set is convex
   - If each constraint gives a convex set, the intersection is a convex set
   - The set $\{x : g_k(x, \theta) \geq 0\}$ is convex as long as $g_k(\cdot, \theta)$ is a quasiconcave function of x
2. The objective function is concave
   - If we only know the objective is quasiconcave, there are other conditions that ensure Kuhn-Tucker is sufficient

Intuition from Kuhn-Tucker conditions I

Recall (evaluating at the optimum, and for all i):

FOC $u_i'(x) = \lambda p_i - \mu_i$

CS $\lambda(w - p \cdot x) = 0$ and $\mu_i x_i = 0$

NN $\lambda \geq 0$ and $\mu_i \geq 0$

Orig $p \cdot x \leq w$ and $x_i \geq 0$

We can summarize as

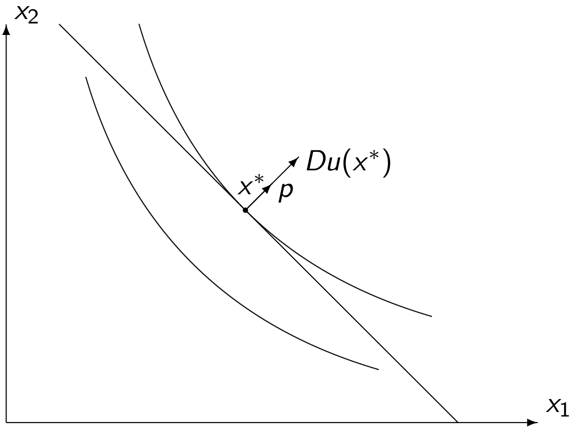
$$u_i'(x) \leq \lambda p_i \text{ with equality if } x_i > 0$$

And therefore if $x_j > 0$ and $x_k > 0$,

$$\frac{p_j}{p_k} = \frac{\frac{\partial u}{\partial x_j}}{\frac{\partial u}{\partial x_k}} \equiv \text{MRS}_{jk}$$

Intuition from Kuhn-Tucker conditions II

- The MRS is the (negative) slope of the indifference curve
- Price ratio is the (negative) slope of the budget line



Intuition from Kuhn-Tucker conditions III

Recall the Envelope Theorem tells us the derivative of the value function in a parameter is the derivative of the Lagrangian:

- Value function (indirect utility)

$$v(p, w) \equiv \sup_{x \in B(p,w)} u(x)$$

- Lagrangian

$$\mathcal{L} \equiv u(x) + \lambda(w - p \cdot x) + \mu \cdot x$$

By the Envelope Theorem, $\frac{\partial v}{\partial w} = \lambda$; i.e., the Lagrange multiplier λ is the "shadow value of wealth" measured in terms of utility

Intuition from Kuhn-Tucker conditions IV

Given our envelope result, we can interpret our earlier condition

$$\frac{\partial u}{\partial x_i} = \lambda p_i \text{ if } x_i > 0$$

as

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial w} p_i \text{ if } x_i > 0$$

where each side gives the marginal utility from an extra unit of x_i

- LHS directly
- RHS through the wealth we could get by selling it

MRS and separable utility

Recall that if $x_j > 0$ and $x_k > 0$,

$$\text{MRS}_{jk} \equiv \frac{\frac{\partial u}{\partial x_j}}{\frac{\partial u}{\partial x_k}}$$

does not depend on λ; however it typically depends on $x_1, \ldots, x_n$

Suppose choice from $X \times Y$ where preferences over X do not depend on y

- Recall that $u(x, y) = U\big(v(x), y\big)$ for some $U(\cdot, \cdot)$ and $v(\cdot)$
- $\frac{\partial u}{\partial x_j} = U_1'\big(v(x), y\big)\frac{\partial v}{\partial x_j}$ and $\frac{\partial u}{\partial x_k} = U_1'\big(v(x), y\big)\frac{\partial v}{\partial x_k}$
- $\text{MRS}_{jk} = \frac{\partial v}{\partial x_j} \big/ \frac{\partial v}{\partial x_k}$ does not depend on y

Separability allows empirical work without worrying about y

Recovering Marshallian demand from indirect utility I

To recover the choice correspondence from the value function we typically apply an Envelope Theorem (e.g., Hotelling, Shephard)

- Value function (indirect utility): $v(p, w) \equiv \sup_{x \in B(p,w)} u(x)$
- Lagrangian: $\mathcal{L} \equiv u(x) + \lambda(w - p \cdot x) + \mu \cdot x$

By the ET

$$\frac{\partial v}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = \lambda$$
$$\frac{\partial v}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i$$

We can combine these, dividing the second by the first...

Recovering Marshallian demand from indirect utility II

Roy's identity

$$x_i(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}}.$$

We can think of this a little bit like "$\frac{\partial v}{\partial w} = -\frac{\partial v}{x_i \partial p_i}$"

Here we showed Roy's identity as an application of the ET; the notes give an entirely different proof that relies on the expenditure minimization problem

Outline

- The utility maximization problem
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  - Comparative statics properties

Why we need another "problem"

We would like to characterize "important" properties of Marshallian demand $x(\cdot,\cdot)$ and indirect utility $v(\cdot,\cdot)$

- Unfortunately, this is harder than doing so for $y(\cdot)$ and $\pi(\cdot)$
- Difficulty arises from the fact that in UMP parameters enter feasible set rather than objective

Consider an price increase for one good (apples)

1. Substitution effect: Apples are now relatively more expensive than bananas, so I buy fewer apples
2. Wealth effect: I feel poorer, so I buy ______ (more? fewer?) apples

Wealth effect and substitution effects could go in opposite directions $\Longrightarrow$ can't easily sign the change in consumption

Isolating the substitution effect

We can isolate the substitution effect by "compensating" the consumer so that her maximized utility does not change

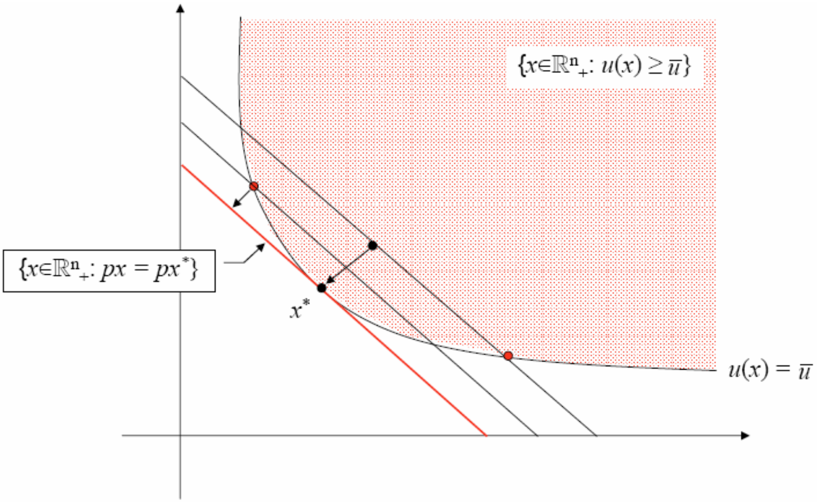
If maximized utility doesn't change, the consumer can't feel richer or poorer; demand changes can therefore be attributed entirely to the substitution effect

Expenditure Minimization Problem

$$\min_{x \in \mathbb{R}^n_+} p \cdot x \text{ such that } u(x) \geq \bar{u}.$$

i.e., find the cheapest bundle at prices p that yield utility at least $\bar{u}$

Illustrating the Expenditure Minimization Problem



Expenditure minimization problem

The consumer's Hicksian demand is given by correspondence $h \colon \mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^n$

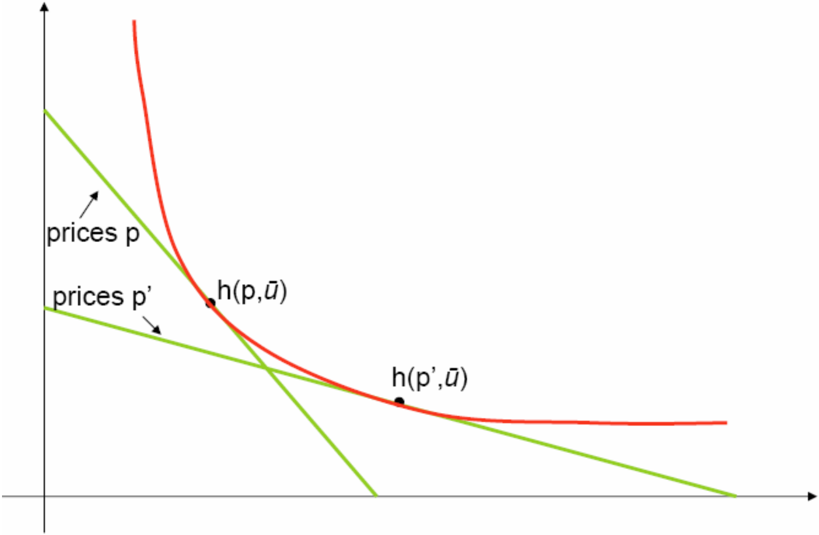
$$\begin{aligned} h(p, \bar{u}) &\equiv \underset{x \in \mathbb{R}^n_+ \colon u(x) \geq \bar{u}}{\operatorname{argmin}} \; p \cdot x \\ &= \{x \in \mathbb{R}^n_+ \colon u(x) \geq \bar{u} \text{ and } p \cdot x = e(p, \bar{u})\} \end{aligned}$$

Resulting expenditure function is given by

$$e(p, \bar{u}) \equiv \min_{x \in \mathbb{R}^n_+ \colon u(x) \geq \bar{u}} p \cdot x$$

Note we have used min instead of inf assuming conditions (listed in the notes) under which a minimum *is* achieved

Illustrating Hicksian demand



Relating Hicksian and Marshallian demand I

Theorem ("Same problem" identities)

Suppose $u(\cdot)$ is a utility function representing a continuous and locally non-satiated preference relation $\succsim$ on $\mathbb{R}^n_+$. Then for any $p \gg \mathbf{0}$ and $w \geq 0$,

1. $h\big(p, v(p, w)\big) = x(p, w)$,
2. $e\big(p, v(p, w)\big) = w$;

and for any $\bar{u} \geq u(\mathbf{0})$,

3. $x\big(p, e(p, \bar{u})\big) = h(p, \bar{u})$, *and*
4. $v\big(p, e(p, \bar{u})\big) = \bar{u}$.

For proofs see notes (cumbersome but relatively straightforward)

Relating Hicksian and Marshallian demand II

These say that UMP and EMP are fundamentally solving the same problem, so:

- If the utility you can get with wealth w is $v(p, w)$...
  - To achieve utility $v(p, w)$ will cost at least w
  - You will buy the same bundle whether you have w to spend, or you are trying to achieve utility $v(p, w)$

- If it costs $e(p, \bar{u})$ to achieve utility $\bar{u}$...
  - Given wealth $e(p, \bar{u})$ you will achieve utility at most $\bar{u}$
  - You will buy the same bundle whether you have $e(p, \bar{u})$ to spend, or you are trying to achieve utility $\bar{u}$

The EMP should look familiar. . .

Expenditure Minimization Problem

$$\min_{x \in \mathbb{R}^n_+} p \cdot x \text{ such that } u(x) \geq \bar{u}.$$

Recall

Single-output Cost Minimization Problem

$$\min_{z \in \mathbb{R}^m_+} w \cdot z \text{ such that } f(z) \geq q.$$

If we interpret $u(\cdot)$ as the production function of the consumer's "hedonic firm," these are the same problem

All of our CMP results go through. . .

Properties of Hicksian demand and expenditure I

As in our discussion of the single-output CMP:

- $e(p, \bar{u}) = p \cdot h(p, \bar{u})$ (adding up)
- $e(\cdot, \bar{u})$ is homogeneous of degree one in p
- $h(\cdot, \bar{u})$ is homogeneous of degree zero in p
- If $e(\cdot, \bar{u})$ is differentiable in p, then $\nabla_p e(p, \bar{u}) = h(p, \bar{u})$ (Shephard's Lemma)
- $e(\cdot, \bar{u})$ is concave in p
- If $h(\cdot, \bar{u})$ is differentiable in p, then the matrix $D_p h(p, \bar{u}) = D_p^2 e(p, \bar{u})$ is symmetric and negative semidefinite
- $e(p, \cdot)$ is nondecreasing in $\bar{u}$
- Rationalizability condition...

Properties of Hicksian demand and expenditure II

Theorem

Hicksian demand function $h\colon P \times \mathbb{R} \rightrightarrows \mathbb{R}^n_+$ *and differentiable expenditure function* $e\colon P \times \mathbb{R} \to \mathbb{R}$ *on an open convex set* $P \subseteq \mathbb{R}^n$ *of prices are jointly rationalizable for a fixed utility* $\bar{u}$ *of a monotone utility function iff*

1. $e(p, \bar{u}) = p \cdot h(p, \bar{u})$ *(adding-up)*;
2. $\nabla_p e(p, \bar{u}) = h(p, \bar{u})$ *(Shephard's Lemma)*;
3. $e(p, \bar{u})$ *is concave in* p *(for a fixed* $\bar{u}$*)*.

The Slutsky Matrix

Definition (Slutsky matrix)

$$D_p h(p, \bar{u}) \equiv \left[\frac{\partial h_i(p, \bar{u})}{\partial p_j}\right]_{i,j} \equiv \begin{bmatrix} \frac{\partial h_1(p,\bar{u})}{\partial p_1} & \cdots & \frac{\partial h_1(p,\bar{u})}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n(p,\bar{u})}{\partial p_1} & \cdots & \frac{\partial h_n(p,\bar{u})}{\partial p_n} \end{bmatrix}.$$

- Concavity of $e(\cdot, \bar{u})$ and Shephard's Lemma give that the Slutsky matrix is symmetric and negative semidefinite (as we found for the substitution matrix)
- $h(\cdot, \bar{u})$ is homogeneous of degree zero in p, so by Euler's Law

$$D_p h(p, \bar{u})\, p = \mathbf{0}$$

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Relating (changes in) Hicksian and Marshallian demand

Assuming differentiability and hence single-valuedness, we can differentiate the ith row of the identity

$$h(p, \bar{u}) = x\big(p, e(p, \bar{u})\big)$$

in p_j to get

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} \underbrace{\frac{\partial e}{\partial p_j}}_{=h_j=x_j}$$

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j$$

The Slutsky equation I

Slutsky equation

$$\underbrace{\frac{\partial x_i(p,w)}{\partial p_j}}_{\text{total effect}} = \underbrace{\frac{\partial h_i\big(p,u(x(p,w))\big)}{\partial p_j}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_i(p,w)}{\partial w}x_j(p,w)}_{\text{wealth effect}}$$

for all i and j.

In matrix form, we can instead write

$$\nabla_p x = \nabla_p h - (\nabla_w x)x^\top.$$

The Slutsky equation II

Setting $i = j$, we can decompose the effect of an an increase in p_i

$$\frac{\partial x_i(p,w)}{\partial p_i} = \frac{\partial h_i\big(p, u(x(p,w))\big)}{\partial p_i} - \frac{\partial x_i(p,w)}{\partial w} x_i(p,w)$$

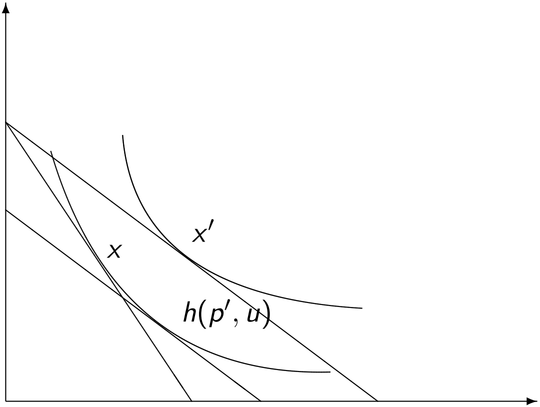
An “own-price” increase. . .

1. Encourages consumer to substitute away from good i
   - $\frac{\partial h_i}{\partial p_i} \leq 0$ by negative semidefiniteness of Slutsky matrix
2. Makes consumer poorer, which affects consumption of good i in some indeterminate way
   - Sign of $\frac{\partial x_i}{\partial w}$ depends on preferences

Illustrating wealth and substitution effects

Following a decrease in the price of the first good...

- Substitution effect moves from x to h
- Wealth effect moves from h to x'



Marshallian response to changes in wealth

Definition (Normal good)

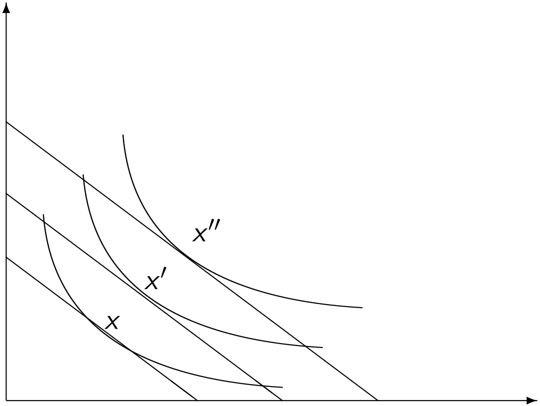
Good i is a normal good if $x_i(p, w)$ is increasing in w.

Definition (Inferior good)

Good i is an inferior good if $x_i(p, w)$ is decreasing in w.

Graphing Marshallian response to changes in wealth

- Engle curves show how Marshallian demand moves with wealth (locus of $\{x, x', x'', \dots\}$ below)
- In this example, both goods are normal (x_i increases in w)



Marshallian response to changes in own price

Definition (Regular good)

Good i is a regular good if $x_i(p, w)$ is decreasing in p_i.

Definition (Giffen good)

Good i is a Giffen good if $x_i(p, w)$ is increasing in p_i.

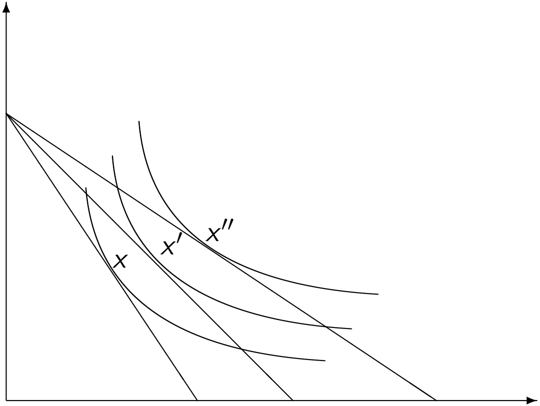
Potatoes during the Irish potato famine are the canonical example (and probably weren't *actually* Giffen goods)

By the Slutsky equation (which gives $\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial w} x_i$ for $i = j$)

- Normal $\implies$ regular
- Giffen $\implies$ inferior

Graphing Marshallian response to changes in own price

- Offer curves show how Marshallian demand moves with price
- In this example, good 1 is regular and good 2 is a gross complement for good 1



Marshallian response to changes in other goods' price

Definition (Gross substitute)

Good i is a gross substitute for good j if $x_i(p, w)$ is increasing in p_j.

Definition (Gross complement)

Good i is a gross complement for good j if $x_i(p, w)$ is decreasing in p_j.

Gross substitutability/complementarity is not necessarily symmetric

Hicksian response to changes in other goods' price

Definition (Substitute)

Good i is a substitute for good j if $h_i(p, \bar{u})$ is increasing in p_j.

Definition (Complement)

Good i is a complement for good j if $h_i(p, \bar{u})$ is decreasing in p_j.

Substitutability/complementarity is symmetric

In a two-good world, the goods must be substitutes (*why?*)

Part VI

Consumer Theory 2

Recap: The consumer problems

Utility Maximization Problem

$$\max_{x \in \mathbb{R}^n_+} u(x) \text{ such that } p \cdot x \leq w.$$

- Choice correspondence: Marshallian demand $x(p, w)$
- Value function: indirect utility function $v(p, w)$

Expenditure Minimization Problem

$$\min_{x \in \mathbb{R}^n_+} p \cdot x \text{ such that } u(x) \geq \bar{u}.$$

- Choice correspondence: Hicksian demand $h(p, \bar{u})$
- Value function: expenditure function $e(p, \bar{u})$

Key questions addressed by consumer theory

Already addressed

- What problems do consumers solve?
- What do we know about the solutions to these CPs generally? What about if we apply restrictions to preferences?
- How do we actually solve these CPs?
- How do the value functions and choice correspondences relate within/across UMP and EMP?

Still to come

- How do we measure consumer welfare?
- How should we calculate price indices?
- When and how can we aggregate across heterogeneous consumers?

Outline

- The welfare impact of price changes
- Price indices
  - Price indices for all goods
  - Price indices for a subset of goods
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Quantifying consumer welfare I

Key question

How much better or worse off is a consumer as a result of a price change from p to p'?

Applies broadly:

- Actual price changes
- Taxes or subsidies
- Introduction of new goods

Quantifying consumer welfare II

Challenge will be to measure how “well off” a consumer is without using utils—recall preference representation is ordinal

This rules out a first attempt:

$$\Delta u = v(p', w) - v(p, w)$$

To get a dollar-denominated measure, we can ask one of two questions:

1. How much would consumer be willing to pay for the price change?
   Fee + Price change $\sim$ Status quo
2. How much would we have to pay consumer to miss out on price change?
   Price change $\sim$ Status quo + Bonus

Quantifying consumer welfare III

Both questions fundamentally ask "how much money is required to achieve a fixed level of utility before and after the price change?"

$$\text{Variation} = e(p, u_{\text{reference}}) - e(p', u_{\text{reference}})$$

For our two questions,

1. How much would consumer be willing to pay for the price change?
   Reference: Old utility ($u_{\text{reference}} = \bar{u} \equiv v(p, w)$)
2. How much would we have to pay consumer to miss out on price change?
   Reference: New utility ($u_{\text{reference}} = \bar{u}' \equiv v(p', w)$)

Compensating and equivalent variation

Definition (Compensating variation)

The amount less wealth (i.e., the fee) a consumer needs to achieve the same maximum utility at new prices (p') as she had before the price change (at prices p):

$$\text{CV} \equiv e\big(p, v(p, w)\big) - e\big(p', v(p, w)\big) = w - e\big(p', \underbrace{v(p, w)}_{\equiv \bar{u}}\big).$$

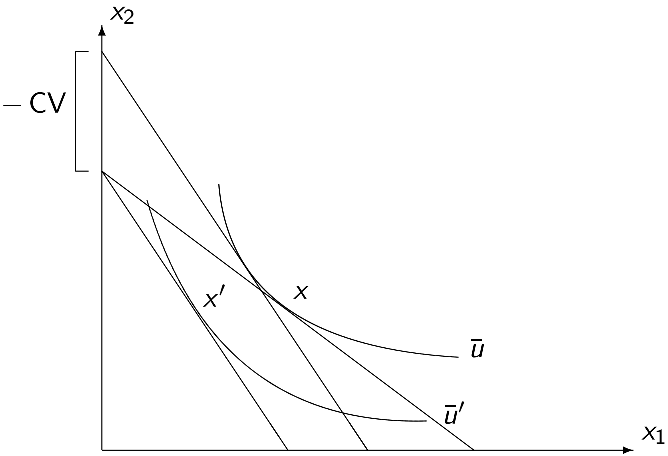
Definition (Equivalent variation)

The amount more wealth (i.e., the bonus) a consumer needs to achieve the same maximum utility at old prices (p) as she could achieve after a price change (to p'):

$$\text{EV} \equiv e\big(p, v(p', w)\big) - e\big(p', v(p', w)\big) = e\big(p, \underbrace{v(p', w)}_{\equiv \bar{u}'}\big) - w.$$

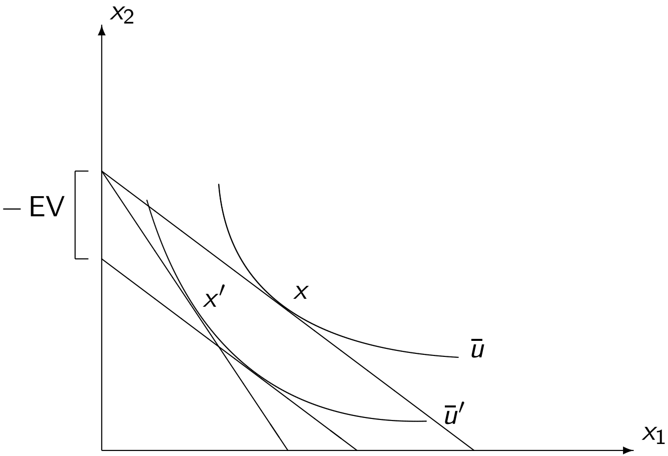
Illustrating compensating variation

- Suppose the price of good two is 1
- Price of good one increases



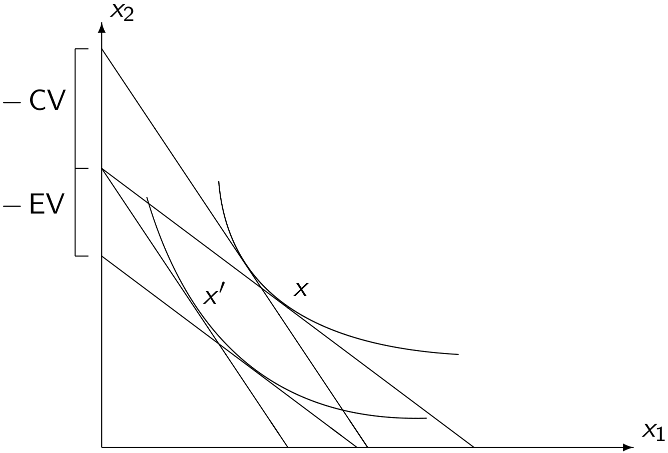
Illustrating equivalent variation

- Suppose the price of good two is 1
- Price of good one increases



We can't order CV and EV

- CV and EV are not necessarily equal
- We can't generally say which is bigger



Changing prices for a single good

Recall

$$\text{CV} = e(p, \bar{u}) - e(p', \bar{u})$$

Suppose the price of a single good changes from $p_i \to p_i'$

$$= \int_{p_i'}^{p_i} \frac{\partial e(p, \bar{u})}{\partial p_i} \, dp_i$$

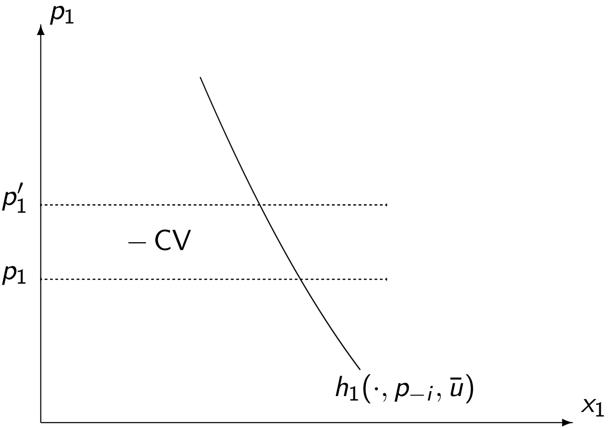
$$= \int_{p_i'}^{p_i} h_i(p, \bar{u}) \, dp_i = - \int_{p_i}^{p_i'} h_i(p, \bar{u}) \, dp_i$$

Similarly,

$$\text{EV} = \int_{p_i'}^{p_i} h_i(p, \bar{u}') \, dp_i = - \int_{p_i}^{p_i'} h_i(p, \bar{u}') \, dp_i$$

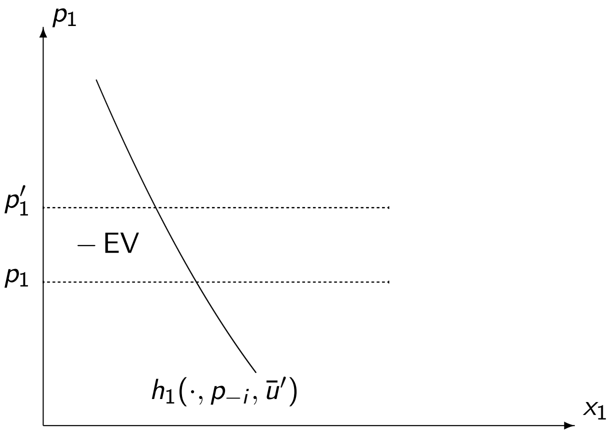
Illustrating changing prices for a single good: CV

- Suppose the price of good one increases from p_1 to p_1'
- Let $\bar{u} \equiv v(p, w)$ and $\bar{u}' \equiv v(p', w)$



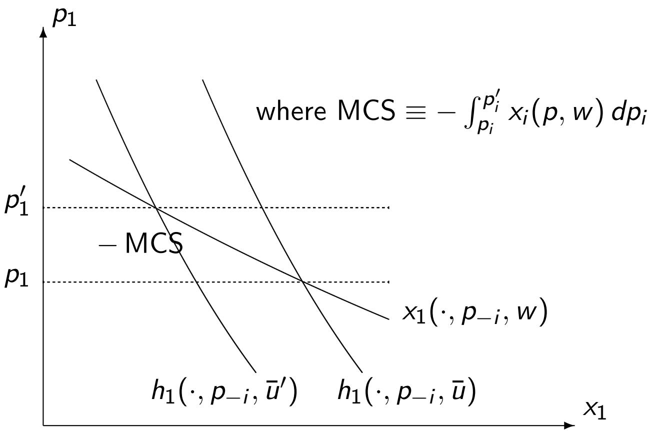
Illustrating changing prices for a single good: EV

- Suppose the price of good one increases from p_1 to p_1'
- Let $\bar{u} \equiv v(p, w)$ and $\bar{u}' \equiv v(p', w)$



Illustrating changing prices for a single good: MCS

- Suppose the price of good one increases from p_1 to p_1'
- Let $\bar{u} \equiv v(p, w)$ and $\bar{u}' \equiv v(p', w)$



Welfare and policy evaluation

- In theory, CV or EV can be summed across consumers to evaluate policy impacts
  - If $\sum_i \text{CV}_i > 0$, we can redistribute from "winners" to "losers," making everyone better off under the policy than before
  - If $\sum_i \text{EV}_i < 0$, we can redistribute from "losers" to "winners," making everyone better off than they would be if policy were implemented
- In reality, identifying winners and losers is difficult
- In reality, widescale redistribution is generally impractical
- Sum-of-CV/EV criterion can cycle (i.e., it can look attractive to enact policy, and then look attractive to cancel it)

Outline

- The welfare impact of price changes
- Price indices
  - Price indices for all goods
  - Price indices for a subset of goods
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Motivation for price indices

Problem: We generally can't access consumers' Hicksian demand correspondences (or even Marshallian ones)

We can say consumers are better off whenever wealth increases more than prices. . . but change of what prices?

1. Ideally we would look at the changing "price" of a "util"
2. Since we can't measure utils, use change in weighted average of goods prices. . . but with what weights?

The Ideal index

The "price" of a "util" is expenditures divided by utility: $\frac{e(p,\bar{u})}{\bar{u}}$

Definition (ideal index)

$$\text{Ideal Index}(\bar{u}) \equiv \frac{p'_{\text{util}}}{p_{\text{util}}} = \frac{e(p',\bar{u})/\bar{u}}{e(p,\bar{u})/\bar{u}} = \frac{e(p',\bar{u})}{e(p,\bar{u})}.$$

Question: what $\bar{u}$ should we use? Natural candidates are

- $v(p,w)$; note $e\big(p, v(p,w)\big) = w$, so denominator equals w
- $v(p',w')$; note $e\big(p', v(p',w')\big) = w'$, so numerator equals w'

Ideal index gives change in wealth required to keep utility constant

Weighted average price indices

We can't measure utility and don't know expenditure function $e(\cdot, \bar{u})$, so settle for an index based on weighted average prices

What weights should we use? Natural candidates are

- Quantity x of goods purchased at old prices p
- Quantity x' of goods purchased at new prices p'

The quantities used to calculated weighted average are often called the "basket"

Defining weighted average price indices

Definition (Laspeyres index)

$$\text{Laspeyres Index} \equiv \frac{p' \cdot x}{p \cdot x} = \frac{p' \cdot x}{w} = \frac{p' \cdot x}{e(p, \bar{u})},$$

where $\bar{u} \equiv v(p, w)$.

Definition (Paasche index)

$$\text{Paasche Index} \equiv \frac{p' \cdot x'}{p \cdot x'} = \frac{w'}{p \cdot x'} = \frac{e(p', \bar{u}')}{p \cdot x'},$$

where $\bar{u}' \equiv v(p', w')$.

Bounding the Laspeyres and Paasche indices

Note that since $u(x) = \bar{u}$ and $u(x') = \bar{u}'$, by "revealed preference"

$$p' \cdot x \geq \min_{\xi:\ u(\xi) \geq \bar{u}} p' \cdot \xi = e(p', \bar{u})$$

$$p \cdot x' \geq \min_{\xi:\ u(\xi) \geq \bar{u}'} p \cdot \xi = e(p, \bar{u}')$$

Thus we get that the Laspeyres index overestimates inflation, while the Paasche index underestimates it:

$$\text{Laspeyres} \equiv \frac{p' \cdot x}{e(p, \bar{u})} \geq \frac{e(p', \bar{u})}{e(p, \bar{u})} \equiv \text{Ideal}(\bar{u})$$

$$\text{Paasche Index} \equiv \frac{e(p', \bar{u}')}{p \cdot x'} \leq \frac{e(p', \bar{u}')}{e(p, \bar{u}')} \equiv \text{Ideal}(\bar{u}')$$

Why the Laspeyres and Paasche indices are not ideal

Deviation of Laspeyres/Paasche indices from Ideal comes from

$$p' \cdot x \geq p' \cdot h(p', \bar{u}) = e(p', \bar{u})$$
$$p \cdot x' \geq p \cdot h(p, \bar{u}') = e(p, \bar{u}')$$

The problem is that

- $p' \cdot x$ doesn't capture consumers' substitution away from x when prices change from p to p'
- $p \cdot x'$ doesn't capture consumers' substitution to x' when prices changed from p to p'

Particular forms of this substitution bias include

- New good bias
- Outlet bias

Price indices for a subset of goods

Suppose we can divide goods into two "groups"

1. Goods E: $\{1, \ldots, k\}$
2. Other goods $\{k+1, \ldots, n\}$

A meaningful price index for E requires that consumers can rank p_E without knowing p_{-E}

For welfare ranking of price vectors for E not to depend on prices for other goods, we must have

$$e(p_E, p_{-E}, \bar{u}) \leq e(p'_E, p_{-E}, \bar{u}) \iff e(p_E, p'_{-E}, \bar{u}') \leq e(p'_E, p'_{-E}, \bar{u}')$$

for all p_E, p'_E, p_{-E}, p'_{-E}, $\bar{u}$, and $\bar{u}'$

A "separability" result for prices

Recall

Theorem

Suppose $\succsim$ on $X \times Y$ is represented by $u(x, y)$. Then preferences over X do not depend on y iff there exist functions $v\colon X \to \mathbb{R}$ and $U\colon \mathbb{R} \times Y \to \mathbb{R}$ such that

1. *$U(\cdot, \cdot)$ is increasing in its first argument, and*
2. *$u(x, y) = U\big(v(x), y\big)$ for all (x, y).*

Theorem

Welfare rankings over p_E do not depend on p_{-E} iff there exist functions $P\colon \mathbb{R}^k \to \mathbb{R}$ and $\hat{e}\colon \mathbb{R} \times \mathbb{R}^{n-k} \times \mathbb{R} \to \mathbb{R}$ such that

1. *$\hat{e}(\cdot, \cdot, \cdot)$ is increasing in its first argument, and*
2. *$e(p, \bar{u}) = \hat{e}\big(P(p_E), p_{-E}, \bar{u}\big)$ for all p and $\bar{u}$.*

Price indices for a subset of goods: other result

Results include that

- This separability in e gives that Hicksian demand for goods outside E only depend on p_E through the price index $P(p_E)$
- $P(\cdot)$ is homothetic (i.e., $P(p'_E) \geq P(p_E) \iff P(\lambda p'_E) \geq P(\lambda p_E)$); we can therefore come up with *some* $P(\cdot)$ which is homogeneous of degree one
- Neither of the two separability conditions defined by the theorems on the previous slide imply each other

More detail is in the lecture notes

Outline

- The welfare impact of price changes
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  - Price indices for a subset of goods
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We can't model the individual consumers in an economy

There are typically too many consumers to model explicitly, so we consider a small number (often only one!)

- Valid if groups of consumers have same preferences and wealth
- If consumers are heterogeneous, validity of aggregation depends on
  - Type of analysis conducted
  - Form of heterogeneity

We consider several forms of analysis: under what forms of heterogeneity can we aggregate consumers?

Types of analysis conducted in the face of heterogeneity

We might try to

1. Model aggregate demand using only aggregate wealth
2. Model aggregate demand using wealth and preferences of a single consumer (i.e., a "positive representative consumer")
3. Model aggregate consumer welfare using welfare of a single consumer (i.e., a "normative representative consumer")

Modelling aggregate demand using aggregate wealth I

Question 1

Can we predict aggregate demand knowing only the aggregate wealth and not its distribution across consumers?

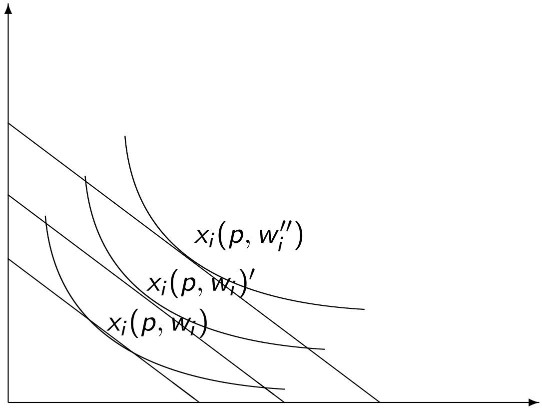
Necessary and sufficient condition: reallocation of wealth never changes total demand; i.e.,

$$\frac{\partial x_i(p, w_i)}{\partial w_i} = \frac{\partial x_j(p, w_j)}{\partial w_j}$$

for all p, i, j, w_i, and w_j

Modelling aggregate demand using aggregate wealth II

- Engle curves must be straight lines, parallel across consumers
- Consumers' indirect utility takes Gorman form:
  $v_i(p, w_i) = a_i(p) + b(p)w_i$



Aggregate demand with positive representative consumer

Question 2

Can aggregate demand be explained as though arising from utility maximization of a single consumer?

Answer: Not necessarily

Aggregate welfare with normative representative consumer

Question 3

Assuming there is a positive representative consumer, can her welfare be used as a proxy for some welfare aggregate of individual consumers?

Answer: Not necessarily

How does this work for firms?

Looking forward to our discussion of general equilibrium, we can also ask about aggregation across firms

Firms aggregate perfectly (assuming price-taking): given J firms,

- Aggregate supply as if single firm with production set

$$Y = Y_1 + \cdots + Y_J = \left\{ \sum_{j=1}^{J} y_j : y_j \in Y_j \text{ for each firm } j \right\}$$

- Profit function $\pi(p) = \sum_j \pi_j(p)$

Firms can aggregate because they have no wealth effects

Outline

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How should consumption be taxed I

Suppose we can impose taxes t in order to fund some spending T
What taxes should we impose? Several ways to approach this

1. Maximize $v(p+t,w)$ such that $t \cdot x(p+t,w) \geq T$
2. Minimize $e(p+t,\bar{u})$ such that $t \cdot h(p+t,\bar{u}) \geq T$

Following the second approach gives Lagrangian

$$\mathcal{L} = -e(p+t,\bar{u}) + \lambda\big(t \cdot h(p+t,\bar{u}) - T\big)$$

And FOC

$$\nabla_p e(p+t^*,\bar{u}) = \lambda h(p+t^*,\bar{u}) + \lambda\big[\nabla_p h(p+t^*,\bar{u})\big] t^*$$

How should consumption be taxed II

$$\underbrace{\nabla_p e(p+t^*,\bar{u})}_{h(p+t^*,\bar{u})} - \lambda h(p+t^*,\bar{u}) = \lambda\left[\nabla_p h(p+t^*,\bar{u})\right] t$$

$$\frac{1-\lambda}{\lambda} h(p+t^*,\bar{u}) = \left[\nabla_p h(p+t^*,\bar{u})\right] t^*$$

$$\frac{1-\lambda}{\lambda}\left[\nabla_p h(p+t^*,\bar{u})\right]^{-1} h(p+t^*,\bar{u}) = t^*$$

This is a generally a difficult system to solve

The no-cross-elasticity case

If $\frac{\partial h_i}{\partial p_j} = 0$ for $i \neq j$, we can solve on a tax-by-tax basis:

$$\lambda t_i^* \frac{\partial h_i(p + t^*, \bar{u})}{\partial p_i} = \underbrace{\frac{\partial e(p + t^*, \bar{u})}{\partial p_i}}_{= h_i(p+t^*, \bar{u})} - \lambda h_i(p + t^*, \bar{u})$$

$$\lambda t_i^* \frac{\partial h_i(p + t^*, \bar{u})}{\partial p_i} = (1 - \lambda) h_i(p + t^*, \bar{u})$$

$$t_i^* = \frac{1 - \lambda}{\lambda} h_i(p + t^*, \bar{u}) \left[\frac{\partial h_i(p + t^*, \bar{u})}{\partial p_i} \right]^{-1}$$

$$\frac{t_i^*}{p_i} = \frac{1 - \lambda}{\lambda} \left[\frac{\partial h_i(p + t^*, \bar{u})}{\partial p_i} \frac{p_i}{h_i(p + t^*, \bar{u})} \right]^{-1}$$

So optimal tax rates are proportional to the inverse of the elasticity of Hicksian demand

Part VII

Choice Under Uncertainty 1

Why study uncertainty?

So far we have covered individual decision-making under certainty

- Goods well understood
- Prices well known

In fact, decisions typically made in the face of an uncertain future

- Workhorse model: objective risk
- Subjective assessments of uncertainty
- Behavioral critiques

von Neumann-Morgenstern expected utility model

Simplifying assumptions include

- Finite number of outcomes (“prizes”)
- Objectively known probability distributions over prizes (“lotteries”)
- Complete and transitive preferences over lotteries
  - Other assumptions on preferences over lotteries (to be discussed)

Outline

- Uncertainty setup
  - Prizes and lotteries
  - Preferences
- Expected utility representation
- Lotteries with monetary payoffs
- Measuring risk aversion
  - Certain equivalent
  - Arrow-Pratt coefficient of absolute risk aversion
  - Risk preferences and wealth

Outline

- Uncertainty setup
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Prizes and lotteries

Let $\mathcal{X}$ be the set of possible prizes (a.k.a. outcomes or consequences)

- Assume $|\mathcal{X}| = n < \infty$
- Since $|\mathcal{X}| < \infty$, there must be a best outcome and a worst outcome

A lottery is a probability distribution over prizes

- $p = (p_1, \dots, p_n) \in \mathbb{R}^n_+$
- The set of all lotteries is

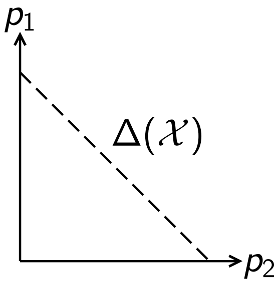
$$\Delta(\mathcal{X}) \equiv \left\{ p \in \mathbb{R}^n_+ : \sum_i p_i = 1 \right\},$$

called the n-dimensional simplex

Graphing the simplex $\Delta(\mathcal{X}) \subseteq \mathbb{R}^2$

Suppose there are two prizes ($|\mathcal{X}| = 2$)

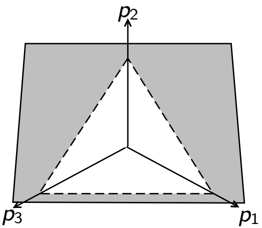
- The simplex $\Delta(\mathcal{X})$ is the portion of the line $p_1 + p_2 = 1$ that lies in the positive quadrant
- This is a one-dimensional submanifold of two-dimensional space—we can draw it as a line segment (i.e., an interval)

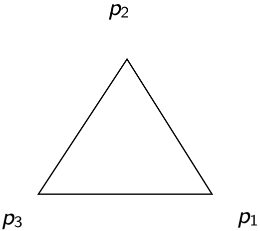


Graphing the simplex $\Delta(\mathcal{X}) \subseteq \mathbb{R}^3$

Suppose there are three prizes ($|\mathcal{X}| = 3$)

- The simplex $\Delta(\mathcal{X})$ is the portion of the plane $p_1 + p_2 + p_3 = 1$ that lies in the positive orthant
- This is a two-dimensional submanifold of three-dimensional space—we can draw it as a triangle





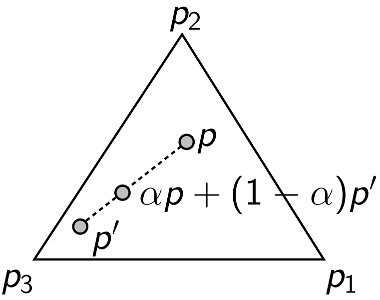
- Aside: These drawings are said to use "barycentric coordinates"

Convexity of the simplex I

Note that $\Delta(\mathcal{X})$ is a convex set

- If $p_i \geq 0$ and $p'_i \geq 0$, then $\alpha p_i + (1-\alpha)p'_i \geq 0$
- If $\sum_i p_i = 1$ and $\sum_i p'_i = 1$, then $\sum_i [\alpha p_i + (1-\alpha)p'_i] = 1$

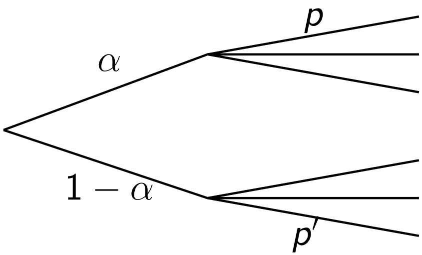
This is not surprising given that the simplex is a “triangle”



Convexity of the simplex II

We can view $\alpha p + (1-\alpha)p'$ as a compound lottery

1. Choose between lotteries: Lottery p with probability α and lottery p' with probability $(1-\alpha)$
2. Resolve uncertainty in chosen lottery per p or p'



Preferences over lotteries

A rational decision-maker has preferences over outcomes $\mathcal{X}$

We consider preferences over lotteries $\Delta(\mathcal{X})$ (note that from here on, $\succsim$ refers to preferences over lotteries, not outcomes)

Expected utility theory relies on $\succsim$ satisfying

- Completeness
- Transitivity
- Continuity (in a sense to be defined)
- Independence (to be defined)

Continuity axiom

Definition (continuity)

A preference relation $\succsim$ over $\Delta(\mathcal{X})$ is continuous iff for any p_H, p_M, and $p_L \in \Delta(\mathcal{X})$ such that $p_H \succsim p_M \succsim p_L$, there exists some $\alpha \in [0,1]$ such that

$$\alpha p_H + (1-\alpha)p_L \sim p_M.$$

Independence axiom

Definition (independence)

A preference relation $\succsim$ over $\Delta(\mathcal{X})$ satisfies independence iff for any p, p', and $p_m \in \Delta(\mathcal{X})$ and any $\alpha \in [0,1]$, we have

$$p \succsim p'$$
$$\Updownarrow$$
$$\alpha p + (1-\alpha)p_m \succsim \alpha p' + (1-\alpha)p_m.$$

i.e., if I prefer p to p', I also prefer the possibility of p to the possibility of p', as long as the other possibility is the same (a $(1-\alpha)$ chance of p_m) in both cases

Independence sensible for choice under uncertainty

There is no counterpart in standard consumer theory; e.g.,

- $p = (2 \text{ coke}, 0 \text{ twinkies})$ and $p' = (0 \text{ coke}, 2 \text{ twinkies})$
- $p_m = (2 \text{ coke}, 2 \text{ twinkies})$
- $\alpha = \frac{1}{2}$

There is no reason to conclude that

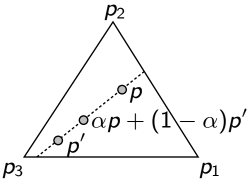
$$\overbrace{(2 \text{ coke}, 0 \text{ twinkies})}^{p} \succsim \overbrace{(0 \text{ coke}, 2 \text{ twinkies})}^{p'}$$
$$\Updownarrow$$
$$\underbrace{(2 \text{ coke}, 1 \text{ twinkies})}_{\alpha p + (1-\alpha) p_m} \succsim \underbrace{(1 \text{ coke}, 2 \text{ twinkies})}_{\alpha p' + (1-\alpha) p_m}$$

Independence implies linear indifference curves

Independence implies linear indifference curves

- Consider $p \sim p'$
- Let $p_m = p'$
- By the independence axiom,

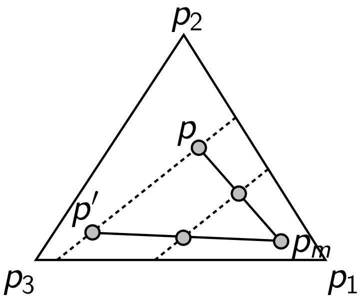
$$\begin{aligned} \alpha p + (1-\alpha)p' &\sim \alpha p' + (1-\alpha)p' \\ &\sim p' \\ &\sim p \end{aligned}$$



Independence implies parallel indifference curves

Independence implies parallel indifference curves

- Consider $p \sim p'$
- Let p_m be some other point, and α some value in $(0,1)$
- By independence $\alpha p + (1-\alpha)p_m \sim \alpha p' + (1-\alpha)p_m$
- $\alpha p + (1-\alpha)p_m$ and $\alpha p' + (1-\alpha)p_m$ lie on a line parallel to the indifference curve containing p and p'



Outline

- Uncertainty setup
  - Prizes and lotteries
  - Preferences
- Expected utility representation
- Lotteries with monetary payoffs
- Measuring risk aversion
  - Certain equivalent
  - Arrow-Pratt coefficient of absolute risk aversion
  - Risk preferences and wealth

von Neumann-Morgenstern utility functions

Definition (von Neumann-Morgenstern utility function)

A utility function $U \colon \Delta(\mathcal{X}) \to \mathbb{R}$ is a vNM utility function iff there exist numbers $u_1, \ldots, u_n \in \mathbb{R}$ such that for every $p \in \Delta(\mathcal{X})$,

$$U(p) = \sum_{i=1}^{n} p_i u_i = p \cdot \vec{u}.$$

Can think of $u_1, \ldots, u_n$ as indexing preference over outcomes

Linearity of vNM utility functions I

Theorem

A utility function $U\colon \Delta(\mathcal{X}) \to \mathbb{R}$ *is a vNM utility function iff it is linear in probabilities, i.e.,*

$$U\big(\alpha p + (1-\alpha)p'\big) = \alpha U(p) + (1-\alpha)U(p')$$

for all p, $p' \in \Delta(\mathcal{X})$, *and* $\alpha \in [0,1]$

Linearity of vNM utility functions II

Proof.

vNM $\implies$ linearity:

$$\begin{aligned} U\big(\alpha p + (1-\alpha)p'\big) &= \big(\alpha p + (1-\alpha)p'\big) \cdot \vec{u} \\ &= (\alpha p) \cdot \vec{u} + \big((1-\alpha)p'\big) \cdot \vec{u} \\ &= \alpha(p \cdot \vec{u}) + (1-\alpha)(p' \cdot \vec{u}) \\ &= \alpha U(p) + (1-\alpha)U(p') \end{aligned}$$

vNM $\impliedby$ linearity: not shown here □

Expected utility representation and ordinality

If preferences $\succsim$ can be represented by a vNM utility function, we say it is an "expected utility representation" of $\succsim$
That is "$U(\cdot)$ is an expected utility representation of $\succsim$" means

1. $U(\cdot)$ is a vNM utility function, and
2. $U(\cdot)$ represents $\succsim$

Linearity of vNM utility functions mean that expected utility representation is not ordinal

- Utility representation is robust to any increasing monotone transformation
- Expected utility representation is only robust to affine (increasing linear) transformations

Exp. util. representation robust to affine transformation I

Theorem

Suppose $U \colon \Delta(\mathcal{X}) \to \mathbb{R}$ *is an expected utility representation of* $\succsim$*. Then* $V \colon \Delta(\mathcal{X}) \to \mathbb{R}$ *is also an expected utility representation of* $\succsim$ *iff there exist some* $a \in \mathbb{R}$ *and* $b \in \mathbb{R}_{++}$ *such that*

$$V(p) = a + bU(p)$$

for all $p \in \Delta(\mathcal{X})$*.*

Exp. util. representation robust to affine transformation II

Proof.

Suppose $U(p) = p \cdot \vec{u}$.

$V(p) = a + bU(p) \Longrightarrow V$ expected utility represents $\succsim$:

- V represents $\succsim$ since $b > 0$ gives that

$$U(p') \geq U(p) \implies a + bU(p') \geq a + bU(p)$$
$$V(p') \geq V(p).$$

- V is vNM utility function since

$$V(p) = a + bU(p) = a + b\sum_{i=1}^{n} p_i u_i = \sum_{i=1}^{n} p_i(\underbrace{a + bu_i}_{\equiv v_i}) = p \cdot \vec{v}.$$

Exp. util. representation robust to affine transformation III

Proof (continued).

$V(p) = a + bU(p) \Longleftarrow V$ expected utility represents $\succsim$:

- Let $\underline{p}$ be the worst lottery (the one giving the worst outcome for certain), and let $\bar{p}$ be the best lottery (the one giving the best outcome for certain). Suppose $\bar{p} \succ \underline{p}$ (if $\bar{p} \sim \underline{p}$ the result is trivial).
- For every $p \in \Delta(\mathcal{X})$, we have $\bar{p} \succsim p \succsim \underline{p}$. Thus $U(\bar{p}) \geq U(p) \geq U(\underline{p})$ so there exists some $\lambda_p \in [0, 1]$ such that

$$U(p) = \lambda_p U(\bar{p}) + (1 - \lambda_p) U(\underline{p}),$$

in particular,

$$\lambda_p = \frac{U(p) - U(\underline{p})}{U(\bar{p}) - U(\underline{p})}.$$

Exp. util. representation robust to affine transformation IV

Proof (continued).

- We have

$$U(p) = \lambda_p U(\bar{p}) + (1-\lambda_p) U(\underline{p})$$
$$= U\big(\lambda_p \bar{p} + (1-\lambda_p)\underline{p}\big)$$
$$p \sim \lambda_p \bar{p} + (1-\lambda_p)\underline{p}.$$

Since V expected utility represents $\succsim$,

$$V(p) = V\big(\lambda_p \bar{p} + (1-\lambda_p)\underline{p}\big)$$
$$= \lambda_p V(\bar{p}) + (1-\lambda_p) V(\underline{p}).$$

Exp. util. representation robust to affine transformation V

Proof (continued).

- Define

$$a \equiv V(\underline{p}) - U(\underline{p})b \quad \text{and} \quad b \equiv \frac{V(\bar{p}) - V(\underline{p})}{U(\bar{p}) - U(\underline{p})}.$$

- We seek to show that $V(p) = a + bU(p)$

$$\begin{aligned} a + bU(p) &= V(\underline{p}) - U(\underline{p})b + bU(p) \\ &= V(\underline{p}) + b\big[U(p) - U(\underline{p})\big] \\ &= V(\underline{p}) + \underbrace{\frac{U(p) - U(\underline{p})}{U(\bar{p}) - U(\underline{p})}}_{\lambda_p}\big[V(\bar{p}) - V(\underline{p})\big] \\ &= \lambda_p V(\bar{p}) + (1 - \lambda_p)V(\underline{p}) \\ &= V(p). \end{aligned}$$

□

Level sets of an expected utility function

A von Neumann-Morgenstern utility function satisfies

$$U(p) = \sum_{i=1}^{n} p_i u_i = p \cdot \vec{u}$$

for some $\vec{u} \in \mathbb{R}^n$

Indifference curves are therefore $p \cdot \vec{u} = c$ for various c

Indifference curves are therefore

- Straight lines ($p \cdot \vec{u} = c$ is a plane that intercepts the simplex in a line)
- Parallel (all indifference curves are normal to $\vec{u}$)

Which preferences have expected utility representations? I

Theorem

A complete and transitive preference relation $\succsim$ on $\Delta(\mathcal{X})$ satisfies continuity and independence iff it has an expected utility representation $U\colon \Delta(\mathcal{X}) \to \mathbb{R}$.

Showing that if $U(p) = p \cdot \vec{u}$ represents $\succsim$, then $\succsim$ must satisfy continuity and independence is (relatively) easy

Showing the other direction is a bit harder...

Which preferences have expected utility representations? II

Formal proof of other direction is given in notes; roughly:

1. For every $p \in \Delta(\mathcal{X})$ find $\lambda_p \in [0,1]$ such that

$$p \sim \lambda_p \bar{p} + (1 - \lambda_p)\underline{p}$$

   This λ_p...

   - exists by continuity
   - is unique by independence

2. Let $U(p) = \lambda_p$

3. Show that $U(\cdot)$ is an expected utility representation of $\succsim$

   - $U(\cdot)$ represents $\succsim$
   - $U(\cdot)$ is linear, and therefore is a vNM utility function

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Money lotteries

We seek to measure the attitude towards risk embedded in preferences $\succsim$

To make things easier, we limit outcomes to monetary payoffs

- $\mathcal{X} \subseteq \mathbb{R}$ (note we give up assumption that prize set is finite)
- $\Delta(\mathcal{X})$ is now a bit more complicated
  - A probability distribution with finite support can be described with a pmf; set of distributions is the simplex
  - A probability distribution over infinite (ordered) support is described by a cdf

From here on, we'll represent a lottery by a cdf $F(\cdot)$, where $F(x)$ is the probability of receiving less than or equal to an \$$x$ payout

What is the set of lotteries?

When $|\mathcal{X}| = n < \infty$, the set of all lotteries is

$$\Delta(\mathcal{X}) \equiv \left\{ p \in \mathbb{R}^n_+ : \sum_i p_i = 1 \right\}$$

When $\mathcal{X} = \mathbb{R}$, the set of all lotteries is the set of cdfs:
$\mathbb{F}$ is the set of all functions $F\colon \mathbb{R} \to [0,1]$ such that

- $F(\cdot)$ is nondecreasing
- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to +\infty} F(x) = 1$
- $F(\cdot)$ is right-continuous (i.e., $\lim_{h \downarrow 0} F(x+h) = F(x)$ for all x)

Preferences over money lotteries

Our old vNM utility function (over pmfs) was

$$U(p) = \sum_i p_i u_i \equiv \mathbb{E}_p u$$

The continuous analogue is a vNM utility function over cdfs:

$$U(F) = \int_{\mathbb{R}} u(x)\, dF(x) \equiv \mathbb{E}_F \left[u(x) \right]$$

Where

- $U \colon \mathbb{F} \to \mathbb{R}$ ("von Neumann-Morgenstern utility function") represents preferences over lotteries
- $u \colon \mathbb{R} \to \mathbb{R}$ ("Bernoulli utility function") indexes preference over outcomes

Risk aversion I

Definition (risk aversion)

A decision-maker is risk-averse iff for all lotteries F, she prefers a certain payoff of $\mathbb{E}_F(x) \equiv \int_{\mathbb{R}} x \, dF(x)$ to the lottery F.

Definition (strict risk aversion)

A decision-maker is strictly risk-averse iff for all non-degenerate lotteries F (i.e, all lotteries for which the support of F is not a singleton), she strictly prefers a certain payoff of $\mathbb{E}_F(x) \equiv \int_{\mathbb{R}} x \, dF(x)$ to the lottery F.

Risk aversion II

Risk aversion says that for all F,

$$u\left(\mathbb{E}_F[x]\right) \geq \mathbb{E}_F\left[u(x)\right]$$

or equivalently

$$u\left(\int_{\mathbb{R}} x \, dF(x)\right) \geq \int_{\mathbb{R}} u(x) \, dF(x)$$

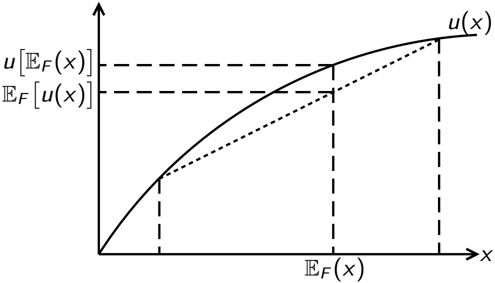
By Jensen's inequality, this condition holds iff $u(\cdot)$ is concave

Theorem

A decision-maker is (strictly) risk-averse iff her Bernoulli utility function is (strictly) concave.

Illustrating risk aversion

Consider a risk-averse decision-maker (i.e., one with a concave Bernoulli utility function) evaluating a lottery F with a two-point distribution



$$u\left(\mathbb{E}_F[x]\right) \geq \mathbb{E}_F\left[u(x)\right]$$

Outline

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The certain equivalent

A risk-averse decision-maker prefers a certain payoff of $\mathbb{E}_F(x)$ to the lottery F

$$u\left(\mathbb{E}_F[x]\right) \geq \mathbb{E}_F\left[u(x)\right]$$

How many "certain" dollars is F worth? That is, what is the certain payoff that gives the same utility as lottery F?

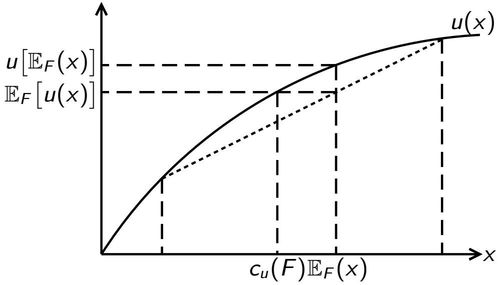
Definition (certain equivalent)

The certain equivalent is the size of the certain payout such that a decision-maker is indifferent between the certain payout and the lottery F:

$$u\left(c_u(F)\right) = \mathbb{E}_F\left[u(x)\right] \equiv \int_{\mathbb{R}} u(x)\, dF(x).$$

Illustrating the certain equivalent

Consider a risk-averse decision-maker (i.e., one with a concave Bernoulli utility function) evaluating a lottery F with a two-point distribution



$$u(c_u(F)) = \mathbb{E}_F\left[u(x)\right] \leq u(\mathbb{E}_F[x])$$

The certain equivalent as a measure of risk aversion

For a risk-averse decision-maker

$$u(c_u(F)) = \mathbb{E}_F\left[u(x)\right] \leq u(\mathbb{E}_F[x])$$

so assuming increasing $u(\cdot)$,

$$c_u(F) \leq \mathbb{E}_F[x]$$

The certain equivalent gives a measure of risk aversion

- A consumer u is risk-averse iff $c_u(F) \leq \mathbb{E}_F[x]$ for all F
- Consumer u is *more* risk-averse than consumer v iff $c_u(F) \leq c_v(F)$ for all F

The Arrow-Pratt coefficient of absolute risk aversion

Definition (Arrow-Pratt coefficient of absolute risk aversion)

Given a twice differentiable Bernoulli utility function $u(\cdot)$,

$$A_u(x) \equiv -\frac{u''(x)}{u'(x)}.$$

Where does this come from?

- Risk-aversion is related to concavity of $u(\cdot)$; a "more concave" function has a smaller second derivative hence a larger $-u''(x)$
  - Normalization by $u'(x)$ takes care of the fact that $au(\cdot) + b$ represents the same preferences as $u(\cdot)$
- We can also view it as a "probability premium"

The A-P coefficient of ARA as a probability premium

Consider a risk-averse consumer:

- She prefers x for certain to a 50-50 gamble between $x + \varepsilon$ and $x - \varepsilon$
- If we wanted to convince her to take such a gamble, it couldn't be 50-50—we need to make the $x + \varepsilon$ payout more likely
- Consider the gamble G such that she is indifferent between G and receiving x for certain, where

$$G = \begin{cases} x + \varepsilon & \text{with probability } \frac{1}{2} + \pi, \\ x - \varepsilon & \text{with probability } \frac{1}{2} - \pi \end{cases}$$

- It turns out that $A_u(x)$ is proportional to (π/ε) as $\varepsilon \to 0$; i.e., $A_u(x)$ tells us the "premium" measured in probability that the decision-maker demands per unit of spread ε

Our measures of risk aversion are equivalent I

Theorem

The following definitions of u being "more risk-averse" than v are equivalent:

1. *Whenever u prefers F to a certain payout d, then v does as well; i.e., for all F and d,*

$$\mathbb{E}_F\left[u(x)\right] \geq u(d) \implies \mathbb{E}_F\left[v(x)\right] \geq v(d);$$

2. *Certain equivalents* $c_u(F) \leq c_v(F)$ *for all F;*
3. $u(\cdot)$ *is "more concave" than* $v(\cdot)$*; i.e., there exists some increasing concave function* $g(\cdot)$ *such that* $u(x) = g\left(v(x)\right)$ *for all x;*
4. *Arrow-Pratt coefficients of absolute risk aversion* $A_u(x) \geq A_v(x)$ *for all x.*

Our measures of risk aversion are equivalent II

Proof.

1 ⇔ 2: Suppose that 1 does not hold; i.e., there exists some F and d such that

$$\begin{aligned} \mathbb{E}_F\big[u(x)\big] &\geq u(d) & \qquad \mathbb{E}_F\big[v(x)\big] &< v(d) \\ u\big[c_u(F)\big] &\geq u(d) & v\big[c_v(F)\big] &< v(d) \\ c_u(F) &\geq d & c_v(F) &< d. \end{aligned}$$

Thus $c_u(F) > c_v(F)$. Implication also goes from bottom to top (if there is some F for which $c_u(F) > c_v(F)$, then there must be some d such that $c_u(F) \geq d > c_v(F)$).

Our measures of risk aversion are equivalent III

Proof (continued).

2 ⇔ 3: u and v are both monotone functions, so there is some increasing function g such that $u(x) = g\big(v(x)\big)$ for all x.

$$c_u(F) \leq c_v(F)$$

$$\Updownarrow$$

$$\begin{aligned} u\big[c_u(F)\big] &\leq u\big[c_v(F)\big] \\ \mathbb{E}_F\big[u(x)\big] &\leq \\ \mathbb{E}_F\big[g\big(v(x)\big)\big] &\leq g\big(v\big[c_v(F)\big]\big) \\ &\leq g\big(\mathbb{E}_F\big[v(x)\big]\big). \end{aligned}$$

This holds for all F iff $g(\cdot)$ is concave by Jensen's inequality.

Our measures of risk aversion are equivalent IV

Proof (continued).

3 ⇔ 4: Existence of $A_u(x)$ and $A_v(x)$ presupposes differentiability.

$$u(x) = g\big(v(x)\big)$$
$$u'(x) = g'\big(v(x)\big)v'(x)$$
$$u''(x) = g'\big(v(x)\big)v''(x) + g''\big(v(x)\big)\big(v'(x)\big)^2$$

$$A_u(x) = -\frac{u''(x)}{u(x)} = -\frac{\cancel{g'\big(v(x)\big)}v''(x)}{\cancel{g'\big(v(x)\big)}v'(x)} - \frac{g''\big(v(x)\big)\big(v'(x)\big)^{\cancel{2}}}{g'\big(v(x)\big)\cancel{v'(x)}}$$
$$= A_v(x) - g''\big(v(x)\big)\frac{v'(x)}{g'\big(v(x)\big)}.$$

Since $v(\cdot)$ and $g(\cdot)$ are increasing functions, we have $A_u(x) \geq A_v(x)$ for all x iff $g'' \leq 0$ for all x. □

How does risk aversion change with "wealth"

Example

Suppose

$$\begin{pmatrix} \$120 & \text{with probability } {}^{2}\!/_{3} \\ \$60 & \text{with probability } {}^{1}\!/_{3} \end{pmatrix} \succsim (\$110 \text{ for certain}).$$

We might then reasonably expect that

$$\begin{pmatrix} \$220 & \text{with probability } {}^{2}\!/_{3} \\ \$160 & \text{with probability } {}^{1}\!/_{3} \end{pmatrix} \succsim (\$210 \text{ for certain}).$$

This is the idea of decreasing absolute risk aversion:
decision-makers are less risk-averse when they are "richer"

Decreasing absolute risk aversion

Definition (decreasing absolute risk aversion)

The Bernoulli utility function $u(\cdot)$ has decreasing absolute risk aversion iff $A_u(\cdot)$ is a decreasing function of x.

Definition (increasing absolute risk aversion)

The Bernoulli utility function $u(\cdot)$ has increasing absolute risk aversion iff $A_u(\cdot)$ is an increasing function of x.

Definition (constant absolute risk aversion)

The Bernoulli utility function $u(\cdot)$ has constant absolute risk aversion iff $A_u(\cdot)$ is a constant function of x.

Relative risk aversion

Definition (coefficient of relative risk aversion)

Given a twice differentiable Bernoulli utility function $u(\cdot)$,

$$R_u(x) \equiv -x\frac{u''(x)}{u'(x)} = xA_u(x).$$

We can define decreasing/increasing/constant relative risk aversion as above, but using $R_u(\cdot)$ instead of $A_u(\cdot)$

- DARA means that if I take a \$10 gamble when poor, I will take a \$10 gamble when rich
- DRRA means that if I gamble 10% of my wealth when poor, I will gamble 10% when rich

Part VIII

Choice Under Uncertainty 2

Outline

- Comparing risky prospects
  - First-order stochastic dominance
  - Second-order stochastic dominance
- Application: demand for insurance
- Application: the portfolio problem
- Beyond the expected utility model
  - Subjective probability
  - Behavioral criticisms

Outline

- Comparing risky prospects
  - First-order stochastic dominance
  - Second-order stochastic dominance
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- Application: the portfolio problem
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  - Subjective probability
  - Behavioral criticisms

When is one lottery "better than" another?

We can compare lotteries given a Bernoulli utility function: $\succsim_u$

When will two lotteries be consistently ranked under a broad set of preferences? e.g.,

1. All nondecreasing $u(\cdot)$
2. All nondecreasing, concave (i.e., risk averse) $u(\cdot)$

Comparing lotteries: examples I

Example

\$95 for certain vs. \$105 for certain.

Example

$$\begin{pmatrix} \$90 & \text{with probability } 1/2 \\ \$110 & \text{with probability } 1/2 \end{pmatrix} \text{ vs. } \$95 \text{ for certain.}$$

Example

$$\begin{pmatrix} \$90 & \text{with probability } 1/2 \\ \$110 & \text{with probability } 1/2 \end{pmatrix} \text{ vs. } \$105 \text{ for certain.}$$

Comparing lotteries: examples II

Example

$$\begin{pmatrix} \$90 & \text{with probability } 1/2 \\ \$110 & \text{with probability } 1/2 \end{pmatrix} \text{ vs. } \$110 \text{ for certain.}$$

Example

$$\begin{pmatrix} \$90 & \text{with probability } 1/2 \\ \$110 & \text{with probability } 1/2 \end{pmatrix} \text{ vs. } \begin{pmatrix} \$80 & \text{with probability } 1/2 \\ \$120 & \text{with probability } 1/2 \end{pmatrix}.$$

First-order stochastic dominance

Definition (first-order stochastic dominance)

Distribution G first-order stochastic dominates distribution F iff lottery G is preferred to F under every nondecreasing Bernoulli utility function $u(\cdot)$.

That is, for every nondecreasing $u\colon \mathbb{R} \to \mathbb{R}$, the following (equivalent) statements hold:

$$G \succsim_u F,$$

$$\mathbb{E}_G\left[u(x)\right] \geq \mathbb{E}_F\left[u(x)\right],$$

$$\int_{\mathbb{R}} u(x)\, dG(x) \geq \int_{\mathbb{R}} u(x)\, dF(x).$$

Characterizing first-order stochastic dominant cdfs I

Theorem

Distribution G first-order stochastic dominates distribution F iff $G(x) \leq F(x)$ for all x.

That is, lottery G is more likely than F to pay at least x for any threshold x

Characterizing first-order stochastic dominant cdfs II

Proof.

We assume differentiability so we can integrate by parts:

$$\int_{\mathbb{R}} u(x)\, dG(x) = u(x)G(x)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} u'(x)G(x)\, dx$$

$$\int_{\mathbb{R}} u(x)\, dF(x) = u(x)F(x)\Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} u'(x)F(x)\, dx.$$

Note that $\lim_{x\to-\infty} u(x)G(x) = \lim_{x\to-\infty} u(x)F(x) = 0$. Assume that $\lim_{x\to+\infty} u(x)[G(x) - F(x)] = 0$, so

$$\int_{\mathbb{R}} u(x)\, dG(x) - \int_{\mathbb{R}} u(x)\, dF(x) = \int_{\mathbb{R}} u'(x)\big[F(x) - G(x)\big]\, dx.$$

Characterizing first-order stochastic dominant cdfs III

Proof (continued).

$$\int_{\mathbb{R}} u(x)\,dG(x) - \int_{\mathbb{R}} u(x)\,dF(x) = \int_{\mathbb{R}} u'(x)\big[F(x) - G(x)\big]\,dx$$

- If $F(x) \geq G(x)$ for all x, the RHS is clearly positive:
  $G(x) \leq F(x)\,\forall x \implies G \succsim_u F\,\forall u(\cdot)$.
- Suppose there is some x' around which $F(x) < G(x)$; we can then consider a $u(\cdot)$ which is constant except in the neighborhood of x'.
  The RHS will therefore be strictly negative, so there exists a nondecreasing $u(\cdot)$ under which $F \succ_u G$:
  $G(x) \leq F(x)\,\forall x \Longleftarrow G \succsim_u F\,\forall u(\cdot)$. □

Back to our examples...

Example

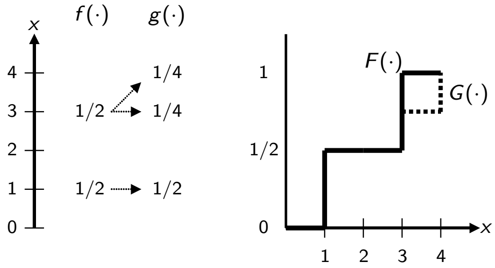
	Lottery				
x	\$95	\$105	\$110	\$80 or \$120	\$90 or \$110
< 80	$F(x) = 0$	0	0	0	0
$[80, 90)$	0	0	0	$1/2$	0
$[90, 95)$	0	0	0	$1/2$	$1/2$
$[95, 105)$	1	0	0	$1/2$	$1/2$
$[105, 110)$	1	1	0	$1/2$	$1/2$
$[110, 120)$	1	1	1	$1/2$	1
≥ 120	1	1	1	1	1

- (\$110) FOSD (\$105) FOSD (\$95)
- (\$110) FOSD (\$90 or \$110)
- Every other combination is ambiguous in terms of FOSD

Characterizing FOSD with upward shifts

Start with a lottery F and construct compound lottery G

- First resolve F
- Then if the resolution of F is some x, hold a second lottery that could potentially increase (but can't decrease) x



G FOSD F iff we can construct G from F using upward shifts

Second-order stochastic dominance

FOSD said a lottery was preferred by *all* nondecreasing $u(\cdot)$...

Consider whether a lottery is preferred by all risk-averse $u(\cdot)$

Definition (second-order stochastic dominance)

Suppose F and G have the same mean.

Distribution G second-order stochastic dominates distribution F iff lottery G is preferred to F under every concave, nondecreasing Bernoulli utility function $u(\cdot)$.

That is, for every concave, nondecreasing $u\colon \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_G\big[u(x)\big] \geq \mathbb{E}_F\big[u(x)\big].$$

Characterizing second-order stochastic dominant cdfs

Theorem

Distribution G second-order stochastic dominates distribution F iff

$$\int_{-\infty}^{x} G(t)\,dt \leq \int_{-\infty}^{x} F(t)\,dt \text{ for all } x.$$

- Proof in notes relies on integration by parts as in FOSD case
- More general proof technique considers basis functions for the class of utility functions
  - Step functions as basis for nondecreasing functions:

$$b_\alpha(x) \equiv \begin{cases} 0, & x \leq \alpha \\ 1, & x > \alpha \end{cases}$$

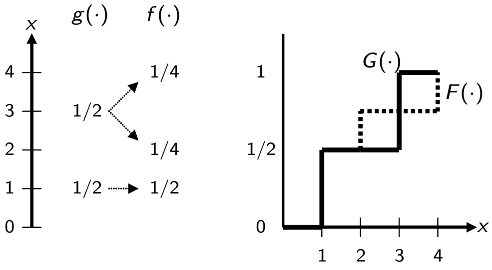
  - Min functions as basis for concave nondecreasing functions:

$$b_\alpha(x) \equiv \min\{x, \alpha\}$$

Characterizing SOSD with mean-preserving spreads

Start with a lottery G and construct compound lottery F

- First resolve G
- Then if the resolution of F is some x, hold a second lottery that adds some *zero-mean* random variable to outcome x



G SOSD F iff we can construct F from G using mean-preserving spreads

Outline

- Comparing risky prospects
  - First-order stochastic dominance
  - Second-order stochastic dominance
- Application: demand for insurance
- Application: the portfolio problem
- Beyond the expected utility model
  - Subjective probability
  - Behavioral criticisms

Demand for insurance: Setup I

- Strictly risk-averse agent with wealth w
- Risk of loss L with probability p
- Insurance available for cost qa pays a in event of loss (agent chooses a)

She solves

$$\max_a pu[w - qa - L + a] + (1 - p)u[w - qa]$$

Demand for insurance: Setup II

The demand for insurance problem

$$\max_a \underbrace{pu[w - qa - L + a] + (1-p)u[w - qa]}_{\equiv U(a)}$$

Note strict concavity of $u(\cdot)$ gives strict concavity of $U(\cdot)$:

$$U'(a) = (1-q)pu'[w - qa - L + a] - q(1-p)u'[w - qa]$$
$$U''(a) = (1-q)^2 pu''[w - qa - L + a] + q^2(1-p)u''[w - qa] < 0$$

Thus FOC is necessary and sufficient:

$$p(1-q)u'[w - qa^* - L + a^*] = q(1-p)u'[w - qa^*]$$

Actuarially fair insurance

What if insurance is actuarially fair?

- That is, insurer makes zero-profit: $q = p$
- FOC becomes

$$\cancel{p(1-q)}u'[w - qa^* - L + a^*] = \cancel{q(1-p)}u'[w - qa^*]$$
$$w - qa^* - L + a^* = w - qa^*$$
$$a^* = L$$

- Agent fully insures against risk of loss

Non-actuarially fair insurance

What if insurance is *not* actuarially fair?

- Suppose cost of insurance is above expected loss: $q > p$
- FOC is

$$\begin{aligned}
\frac{u'[w - qa^* - L + a^*]}{u'[w - qa^*]} &= \frac{q(1-p)}{p(1-q)} \\
&> 1 \\
u'[w - qa^* - L + a^*] &> u'[w - qa^*] \\
w - qa^* - L + a^* &< \quad w - qa^* \\
a^* &< L
\end{aligned}$$

- Agent under-insures against risk of loss; it's costly to transfer wealth to the loss state, so she transfers less

A useful comparative statics result I

Consider maximizing a strictly concave differentiable function $U(\cdot)$

FOC: $U'(x^*) = 0$

Note that since $U(\cdot)$ is concave, $U'(\cdot)$ is decreasing

- $U'(x) > 0 = U'(x^*) \iff x < x^*$ (i.e., $U(\cdot)$ is increasing to the left of the maximum)
- $U'(x) < 0 = U'(x^*) \iff x > x^*$ (i.e., $U(\cdot)$ is decreasing to the right of the maximum)

A useful comparative statics result II

Now consider maximizing $U(\cdot, w)$ given a parameter w

FOC: $\partial_x U\big(x^*(w), w\big) = 0$

As above, we know $x < x^*(w)$ iff

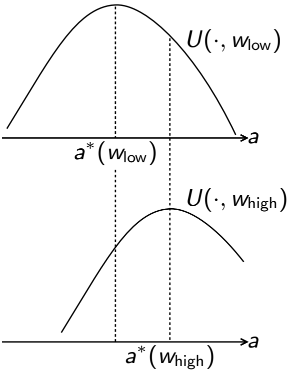
$$\partial_x U(x, w) > 0$$

Letting $x = x^*(w_l)$ for some w_l gives $x^*(w_l) < x^*(w_h)$ iff

$$\partial_x U\big(x^*(w_l), w_h\big) > 0 = \partial_x U\big(x^*(w_l), w_l\big)$$

Suppose the cross partial $\partial_w \partial_x U\big(x^*(w_l), \cdot\big) > 0$; then for all $w_h > w_l$ the above condition holds so $x_*(w_h) > x_*(w_l)$

A useful comparative statics result III



FOCs:

- $\partial_a U\big(a^*(w_{\text{low}}), w_{\text{low}}\big) = 0$
- $\partial_a U\big(a^*(w_{\text{high}}), w_{\text{high}}\big) = 0$

$a^*(w_{\text{low}}) < a^*(w_{\text{high}})$ implies

- $\partial_a U\big(a^*(w_{\text{low}}), w_{\text{high}}\big) > 0$
- $\partial_a U\big(a^*(w_{\text{high}}), w_{\text{low}}\big) < 0$

Sufficient condition is that

$$\partial_w \partial_a U\big(a^*(w_l), \cdot\big) > 0$$

Towards comparative statics in wealth

We showed already that given strict risk-aversion,

- If insurance is actuarially fair ($q = p$), agents fully insure
- If cost of insurance is above expected loss ($q > p$), agents under-insure

Intuitively, an agent with decreasing absolute risk aversion will insure less when wealth is higher

$$\begin{aligned} U(a,w) &= pu[w - qa - L + a] + (1-p)u[w - qa] \\ \partial_w U(a,w) &= pu'[w - qa - L + a] + (1-p)u'[w - qa] \\ \partial_a \partial_w U(a,w) &= p(1-q)u''[w - qa - L + a] - q(1-p)u''[w - qa] \end{aligned}$$

We can't sign this (and use Topkis), but if we can sign it at $a^*(w)$ we can use our previous result...

Towards comparative statics in wealth I

Recall the FOC:

$$p(1-q)u'[\underbrace{w - qa^* - L + a^*}_{\equiv b^*}] = q(1-p)u'[\underbrace{w - qa^*}_{\equiv g^*}]$$

$$\frac{p(1-q)u'[b^*]}{u'[g^*]} = q(1-p)$$

Where b^* is the payout in the "bad" (loss) state, and g^* the payout in the "good" state

By our earlier result, agents under-insure: $g^* > b^*$

Towards comparative statics in wealth II

We can plug in to our earlier expression to get

$$\begin{aligned}
\partial_a \partial_w U(a^*, w) &= p(1-q)u''[b^*] - q(1-p)u''[g^*] \\
&= p(1-q)u'[b^*]\left[\frac{u''[b^*]}{u'[b^*]} - \frac{u''[g^*]}{u'[g^*]}\right] \\
&= p(1-q)u'[b^*]\underbrace{\left[-A[b^*] + A[g^*]\right]}_{<0 \text{ by DARA}} \\
&< 0
\end{aligned}$$

By our "useful comparative statics result,"

When $p < q$, the agent will under-insure; $a^*(w)$ is decreasing in wealth if the agent has DARA and increasing in wealth if the agent has IARA.

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The portfolio problem: Setup

- Risk-averse agent with wealth w
- Choice of investment across two assets
  1. Safe asset returns r for certain (investment $w - a$)
  2. Risky asset returns z distributed according to cdf $F(\cdot)$, with a higher expected return than the safe asset; i.e., $\mathbb{E}_F z > r$ (investment a)

Agent winds up with $az + (w - a)r$

She solves

$$\max_a \underbrace{\mathbb{E}_F\left[u(az + (w - a)r)\right]}_{U(a)} \equiv \max_a \underbrace{\int_{\mathbb{R}} u\big(az + (w - a)r\big)\, dF(z)}_{U(a)}$$

First-order condition:

$$\int_{\mathbb{R}} (z - r)u'\big(a^* z + (w - a^*)r\big)\, dF(z) = 0$$

Risk-neutral agent

If agent is risk-neutral, $u(x) = \alpha x + \beta$; the FOC would require

$$\int_{\mathbb{R}} (z - r) u'\big(a^* z + (w - a^*) r\big)\, dF(z) \overset{?}{=} 0$$

$$\int_{\mathbb{R}} (z - r) \alpha\, dF(z) \overset{?}{=}$$

$$\int_{\mathbb{R}} (z - r)\, dF(z) \overset{?}{=}$$

$$\mathbb{E}_F z \neq r$$

We have a corner solution: agent puts all investment in risky asset

A risk-neutral investor only cares about *expected* return

Strictly risk-averse agent: intuition from the FOC

If agent is strictly risk-averse, strict concavity of $u(\cdot)$ gives strict concavity of $U(\cdot)$: FOC is necessary and sufficient

$$\int_{\mathbb{R}} (z - r) u'\big(a^* z + (w - a^*) r\big)\, dF(z) = 0$$

Could $a^* = 0$ be a solution? Only if

$$\int_{\mathbb{R}} (z - r) u'(wr)\, dF(z) \stackrel{?}{=} 0$$

$$\int_{\mathbb{R}} (z - r)\, dF(z) \stackrel{?}{=} 0$$

$$\mathbb{E}_F z \neq r$$

A strictly risk-averse investor will always invest at least a little in the risky asset!

A useful comparative statics result

Recall our earlier result: when maximizing a strictly concave differentiable function $U(\cdot)$,

- $U'(a) > 0 = U'(a^*) \iff a < a^*$ (i.e., $U(\cdot)$ is increasing to the left of the maximum)
- $U'(a) < 0 = U'(a^*) \iff a > a^*$ (i.e., $U(\cdot)$ is decreasing to the right of the maximum)

Therefore if $U'(a) = 0 \implies V'(a) \geq 0$, we know that

$$\operatorname*{argmax}_a U(a) \leq \operatorname*{argmax}_a V(a).$$

Comparative statics in risk aversion I

Suppose agent v is less risk-averse than agent u

- We expect v to invest more in the risky asset
- It is a sufficient condition for $U'(a) = 0 \implies V'(a) \geq 0$

We can write $v(x) = h\big(u(x)\big)$ for some nondecreasing *convex* $h(\cdot)$

$$U(a) \equiv \int_{\mathbb{R}} u\big(az + (w - a)r\big)\, dF(z)$$

$$V(a) \equiv \int_{\mathbb{R}} v\big(az + (w - a)r\big)\, dF(z)$$

$$= \int_{\mathbb{R}} h\Big[u\big(az + (w - a)r\big)\Big]\, dF(z)$$

$$V'(a) = \int_{\mathbb{R}} h'\Big[u\big(az + (w - a)r\big)\Big] u'\big(az + (w - a)r\big)(z - r)\, dF(z)$$

Comparative statics in risk aversion II

Consider evaluating $V'(a^*)$ where a^* satisfies $U'(a^*) = 0$
Consider the term $h'\left[u\left(a^* z + (w - a^*)r\right)\right]$, which is increasing in z since $h(\cdot)$ is convex

- Define $\tilde{h}$ as this term evaluated when $z = r$; note $\tilde{h} \geq 0$
- When $z \leq r$, the term is below $\tilde{h}$
- When $z \geq r$, the term is above $\tilde{h}$

Comparative statics in risk aversion III

$$V'(a^*) = \int_{\mathbb{R}} h'\big[u\big(a^*z+(w-a^*)r\big)\big]\,u'\big(a^*z+(w-a^*)r\big)(z-r)\,dF(z)$$

$$= \int_{-\infty}^{r} \underbrace{h'\big[u\big(a^*z+(w-a^*)r\big)\big]}_{\leq \tilde{h}}\,u'\big(a^*z+(w-a^*)r\big)\underbrace{(z-r)}_{\leq 0}\,dF(z)$$

$$+ \int_{r}^{\infty} \underbrace{h'\big[u\big(a^*z+(w-a^*)r\big)\big]}_{\geq \tilde{h}}\,u'\big(a^*z+(w-a^*)r\big)\underbrace{(z-r)}_{\geq 0}\,dF(z)$$

$$\geq \int_{-\infty}^{r} \tilde{h}u'\big(a^*z+(w-a^*)r\big)(z-r)\,dF(z)$$

$$+ \int_{r}^{\infty} \tilde{h}u'\big(a^*z+(w-a^*)r\big)(z-r)\,dF(z)$$

$$= \int_{\mathbb{R}} \tilde{h}u'\big(a^*z+(w-a^*)r\big)(z-r)\,dF(z) = \tilde{h}U'(a^*) = 0$$

Comparative statics in risk aversion: what have we done?

- We showed that $U'(a) = 0 \implies V'(a) \geq 0$
- Therefore $\text{argmax}_a U(a) \leq \text{argmax}_a V(a)$
- This confirms our "intuition" that a less risk-averse agent invests more in the risky asset

The notes use the same logic to argue that an agent with DARA will invest more in the risky asset at higher levels of wealth

Similar results can be proven for DRRA

Outline

- Comparing risky prospects
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  - Second-order stochastic dominance
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Subjective probabilities

We have assumed the decision-maker accurately understands the likelihood of each outcome; suppose not

- Set of outcomes $\mathcal{X}$ (e.g., dollar payouts)
- Set of "states of the world" $\mathcal{S}$
- Bets (a.k.a. "acts") are mappings from states of the world to outcomes $f : \mathcal{S} \to \mathcal{X}$

Savage shows that if preferences over acts satisfy certain axioms, they must be *as if* maximizing a vN-M utility function given

- Some probability distribution over $\mathcal{S}$
- A Bernoulli utility function $u : \mathcal{X} \to \mathbb{R}$

Problems with the independence axiom

Example

Which lottery would you rather face?

	$0	$48	$55	Expected payout
Lottery A	1%	66%	33%	$50
Lottery B		100%		$48

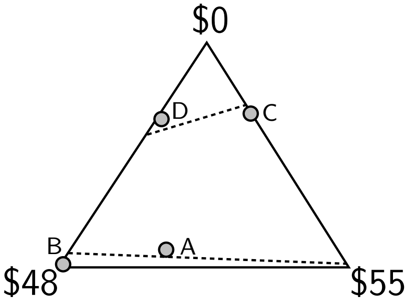
Example

Which lottery would you rather face?

	$0	$48	$55	Expected payout
Lottery C	67%		33%	$18
Lottery D	66%	34%		$16

Illustrating Allais' experiment

	\$0	\$48	\$55	Expected payout
Lottery A	1%	66%	33%	\$50
Lottery B		100%		\$48
Lottery C	67%		33%	\$18
Lottery D	66%	34%		\$16



Problems with risk aversion

It seems like people are way too risk averse on small-stakes gambles

Example

Would you bet on a fair coin toss where you lose $1000 or win $1050?

If you would always turn down such a bet (at any wealth level), you would turn down a bet on a fair coin where you lose $20,000 or gain *any* amount

"Loss aversion" has been suggested as an explanation

Framing experiment 1

Example

The U.S. is preparing for an outbreak of an unusual disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Scientists predict that:

- If program A is adopted, 200 people will be saved. [72%]
- If program B is adopted, there is a $^2/_3$ chance that no one will be saved, and a $^1/_3$ probability that 600 people will be saved. [28%]

Which program would you choose?

Framing experiment 2

Example

The U.S. is preparing for an outbreak of an unusual disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Scientists predict that:

- If program C is adopted, 400 people will die with certainty. [22%]
- If program D is adopted, there is a $^2/_3$ probability that 600 people will die, and a $^1/_3$ probability that no one will die. [78%]

Which program would you choose?

These two examples are exactly the same question stated in different ways!

Part IX

General Equilibrium 1

What is general equilibrium?

So far we have talked about producers and consumers

- One producer
- One consumer
- Several consumers (aggregation can be tricky)
- Several producers (aggregation straightforward)

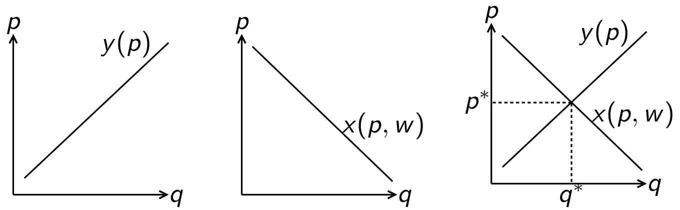
Main shortcoming was not that we only looked at consumers or producers, but rather that we treated prices as exogenous

Why does demand equal supply?

Simple story

- Quantity produced as function of price (producer theory)
- Quantity consumed as function of price (consumer theory)

Two equations in two unknowns give a solution



Actually, the story is a bit more complicated. . .

Supply and demand for each good depend on prices of other goods

- Supply $\vec{y}(\vec{p})$
- Marshallian demand $\vec{x}(\vec{p}, w)$

General equilibrium prices satisfy

$$\vec{y}(\vec{p}) = \vec{x}(\vec{p}, w),$$

potentially a very complicated system of equations

General equilibrium: key questions

- Does a general equilibrium exist?
  - Uniqueness
  - "Stability"
- If so, what are it's properties? In particular, in what ways is it "efficient"?
- How does the economy reach general equilibrium prices?

An important simplification

It turns out that finding prices that equalize production and demand is a hard problem

Initially we will ignore production: exchange economy

- Finite number of agents
- Finite number of goods
- Predetermined amount of each commodity (no production)
- Goods get traded and consumed

Other assumptions

None of the following assumptions should surprise at this point, but should be kept in mind when interpreting our following results:

- Markets exist for all goods
- Agents can freely participate in markets without cost
- "Standard" consumer theory assumptions
    - Preferences can be represented by a utility function
    - Preferences are LNS/monotone/strictly monotone (as needed)
    - All agents are price takers
    - Finite number of divisible goods
    - Linear prices
    - Perfect information about goods and prices
- All agents face the same prices

Outline

- Exchange economies: the Walrasian Model
  - Walrasian equilibrium
  - Pareto optimality
- The Welfare theorems
- Characterizing optimality and equilibrium with first-order conditions
  - The Pareto problem
  - The Walrasian problem

Outline

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Exchange economies: the Walrasian Model

Primitives of the model

- L goods $\ell \in \mathcal{L} \equiv \{1, \dots, L\}$
- I agents $i \in \mathcal{I} \equiv \{1, \dots, I\}$
  - Endowments $e^i \in \mathbb{R}^L_+$; agents do not have monetary wealth, but rather an endowment of goods which they can trade or consume
  - Preferences represented by utility function $u^i \colon \mathbb{R}^L_+ \to \mathbb{R}$
- Endogenous prices $p \in \mathbb{R}^L_+$, taken as given by each agent

Each agent i solves

$$\max_{x^i \in \mathbb{R}^L_+} u^i(x^i) \text{ such that } p \cdot x^i \leq p \cdot e^i \equiv \max_{x^i \in B^i(p)} u^i(x^i)$$

where $B^i(p) \equiv \{x^i \in \mathbb{R}^L_+ : p \cdot x^i \leq p \cdot e^i\}$ is the budget set for i

Walrasian equilibrium

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium iff

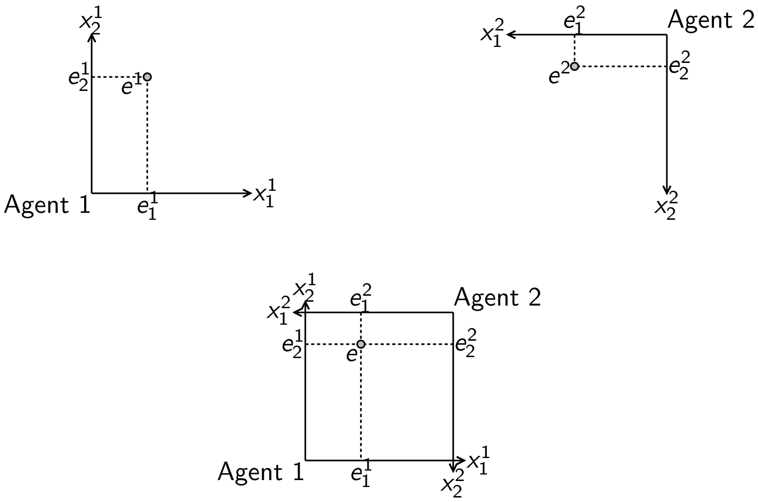
1. All agents maximize their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$

2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

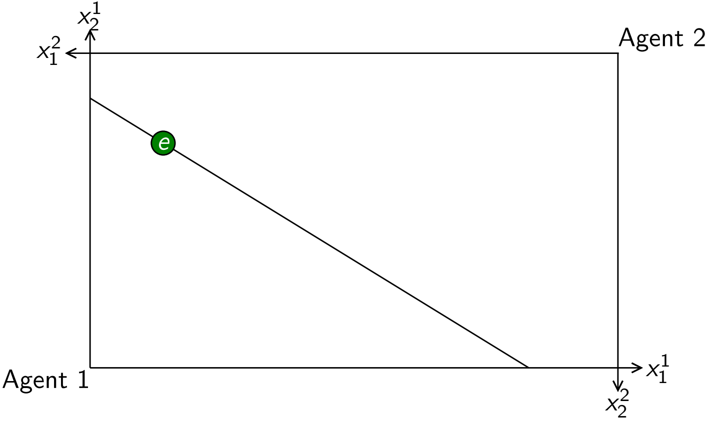
$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell.$$

A graphical example: the Edgeworth box



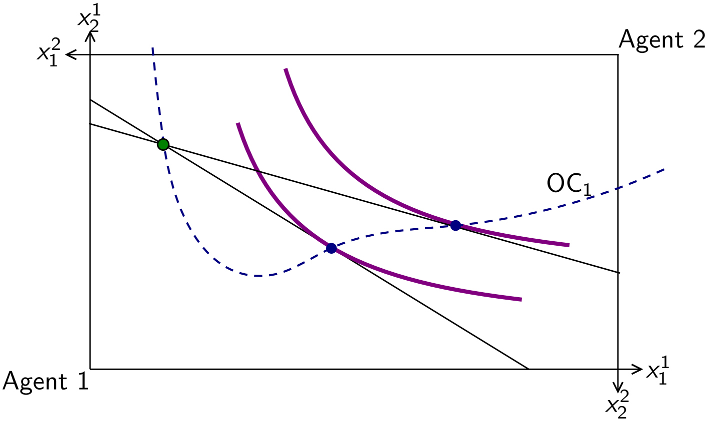
A graphical example: the Edgeworth box

$$p \cdot x^1 = p \cdot e^1 \text{ coincides with } p \cdot x^2 = p \cdot e^2$$

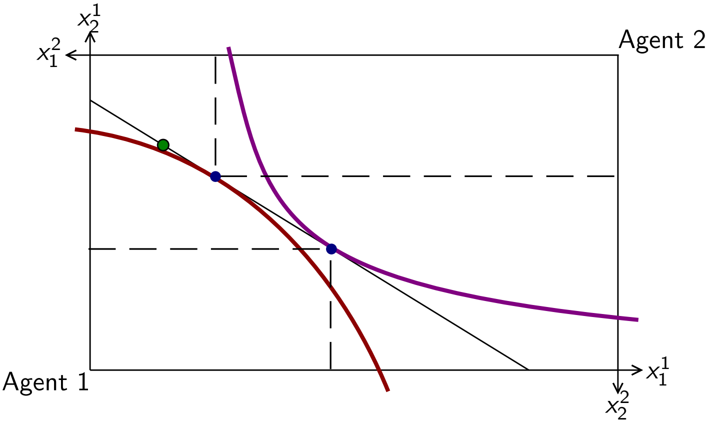


The offer curve

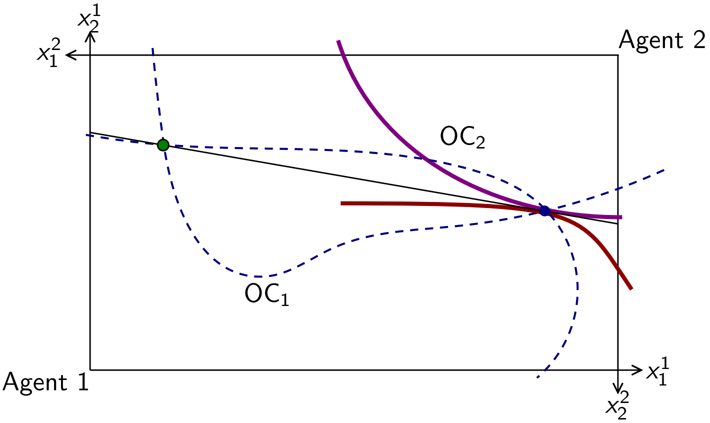
The offer curve traces out Marshallian demand as prices change



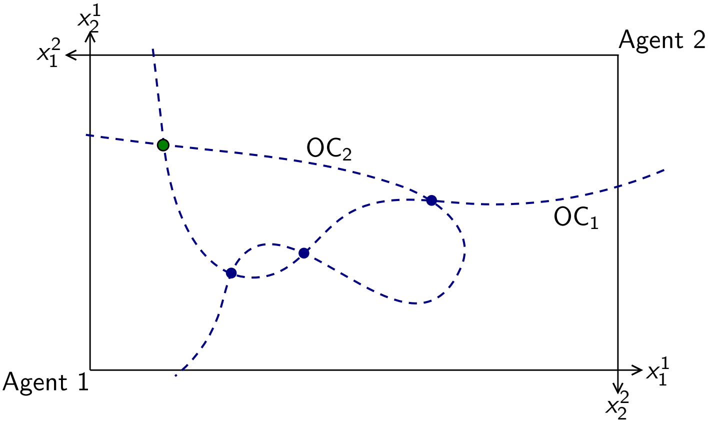
Non-equilibrium prices give total demand ≠ supply



Walrasian equilibria are at the intersection of offer curves



There may be a multiplicity of Walrasian equilibria



Pareto optimality

Definition (feasible allocation)

Allocations $(x^i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{I \cdot L}$ are feasible iff for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i.$$

Definition (Pareto optimality)

Allocations $x \equiv (x^i)_{i \in \mathcal{I}}$ are Pareto optimal iff

1. x is feasible, and
2. There is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

Pareto optimality: General case

More generally, Pareto optimality can be defined over outcomes $\mathcal{X}$ for any set of agents $\mathcal{I}$ given notions of

1. Feasibility: a mapping $\mathcal{X} \to \{\text{infeasible}, \text{feasible}\}$
2. Individual preferences: rational preferences $\succsim^i$ over $\mathcal{X}$ for each $i \in \mathcal{I}$

Definition (Pareto optimality)

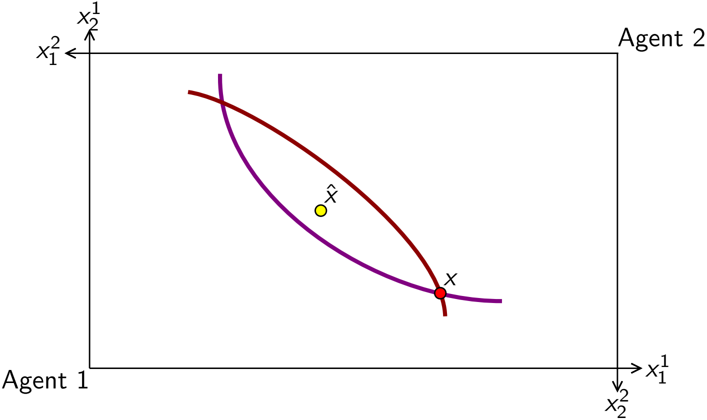
Outcome $x \in \mathcal{X}$ is Pareto optimal iff

1. x is feasible, and
2. There is no other feasible outcome $\hat{x} \in \mathcal{X}$ such that $\hat{x} \succsim^i x$ for all $i \in \mathcal{I}$ with $\hat{x} \succ^i x$ for some i.

This is a very weak notion of optimality, requiring only that there is nothing "left on the table"

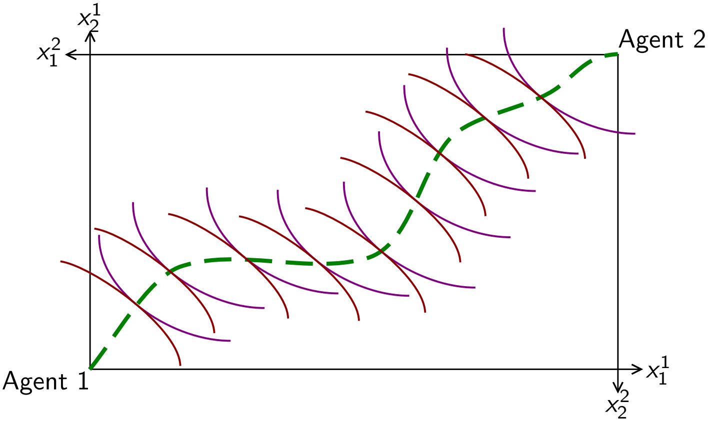
Pareto optimality in the Edgeworth box

If the indifference curves passing through x are not tangent, it is not Pareto optimal



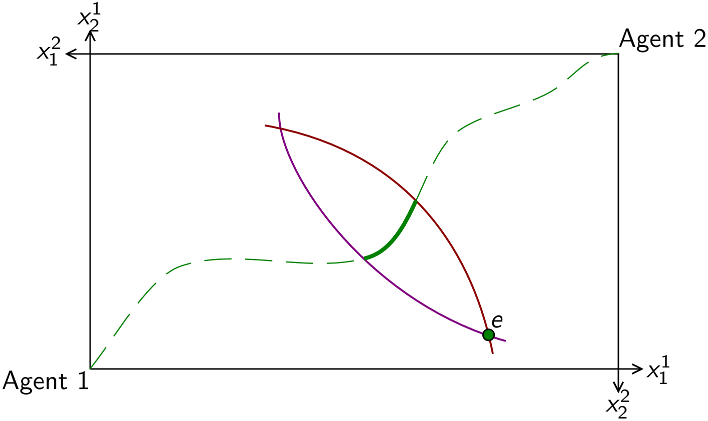
The Pareto set

The Pareto set is the locus of Pareto optimal allocations



The contract curve

We expect agents to reach the contract curve: the portion of the Pareto set that makes each better off than e



Outline

- Exchange economies: the Walrasian Model
  - Walrasian equilibrium
  - Pareto optimality
- The Welfare theorems
- Characterizing optimality and equilibrium with first-order conditions
  - The Pareto problem
  - The Walrasian problem

Relating Walrasian equilibrium and Pareto optimality

Note Walrasian Equilibria and Pareto Optima are very different concepts

Pareto optimality

1. Allocations

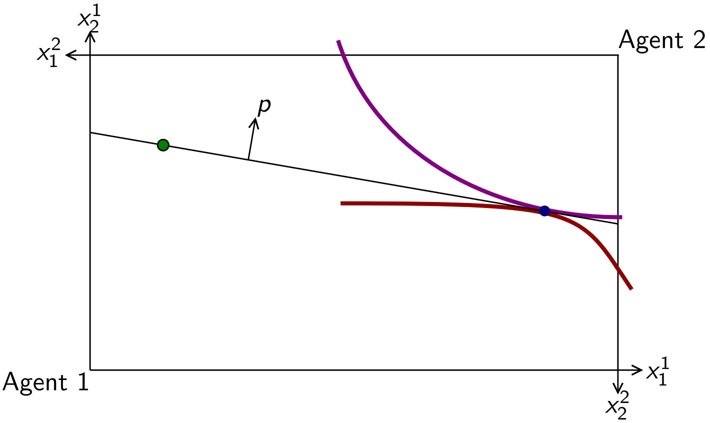
given total endowments and individual preferences.

Walrasian equilibrium

1. Allocations
2. Prices

given individual endowments and preferences.

Walrasian equilibrium allocations are Pareto optimal



The First Welfare Theorem: WE are PO I

Theorem (First Welfare Theorem)

Suppose $u^i(\cdot)$ is increasing (i.e., $u^i(x^{i\prime}) > u^i(x^i)$ for any $x^{i\prime} \gg x^i$) for all $i \in \mathcal{I}$.
If p and $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium, then the allocations $(x^i)_{i \in \mathcal{I}}$ are Pareto optimal.

Proof.

Suppose in contradiction that $\hat{x}$ Pareto dominates x; i.e.,

1. $\hat{x}$ is feasible,
2. $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$,
3. $u^i(\hat{x}^{i'}) > u^{i'}(x^{i'})$ for some $i' \in \mathcal{I}$.

The First Welfare Theorem: WE are PO II

Proof (continued).

By revealed preference and Walras' law, $p \cdot \hat{x}^i \geq p \cdot x^i$ for all i, and $p \cdot \hat{x}^{i'} > p \cdot x^{i'}$. Thus

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}^i > \sum_{i \in \mathcal{I}} p \cdot x^i$$

$$\sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{I}} p_\ell \hat{x}^i_\ell > \sum_{\ell \in \mathcal{L}} \sum_{i \in \mathcal{I}} p_\ell x^i_\ell.$$

So for some $\widetilde{\ell}$ it must be that

$$\sum_{i \in \mathcal{I}} \hat{x}^i_{\tilde{\ell}} > \sum_{i \in \mathcal{I}} x^i_{\tilde{\ell}} = \sum_{i \in \mathcal{I}} e^i_{\tilde{\ell}},$$

so $\hat{x}$ cannot be feasible. □

The Second Welfare Theorem: PO endowments are WE

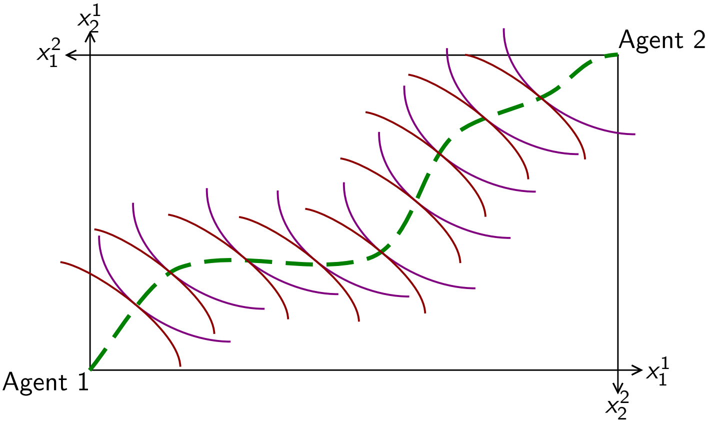
Theorem (Second Welfare Theorem)

Suppose for all $i \in \mathcal{I}$,

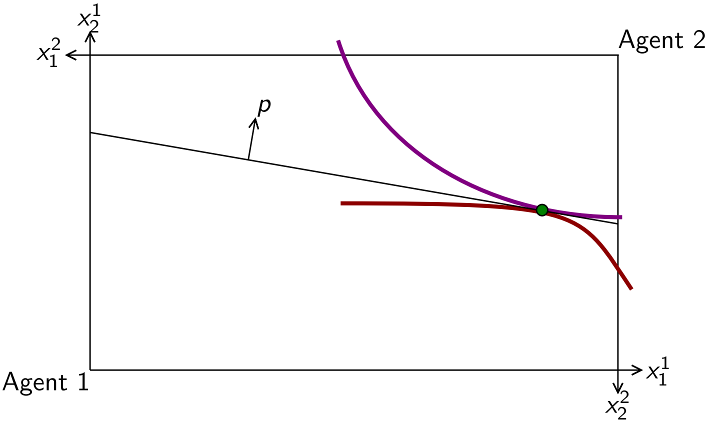
1. $u^i(\cdot)$ *is continuous;*
2. $u^i(\cdot)$ *is increasing; i.e.,* $u^i(x^{i\prime}) > u^i(x^i)$ *for any* $x^{i\prime} \gg x^i$;
3. $u^i(\cdot)$ *is concave; and*
4. $e^i \gg \mathbf{0}$; *i.e., every agent has at least a little bit of every good.*

If $(e^i)_{i \in \mathcal{I}}$ *are Pareto optimal, then there exist prices* $p \in \mathbb{R}^l_+$ *such that* p *and* $(e^i)_{i \in \mathcal{I}}$ *are a Walrasian equilibrium.*

Pareto optimal allocations can be supported as WE. . .



. . . with prices that separates agents' upper contour sets



Return of the Separating Hyperplane Theorem

Theorem (Separating Hyperplane Theorem)

Suppose that $A \subseteq \mathbb{R}^n$ is an open, convex set and that $x \notin A$. Then there exists $\theta \neq \mathbf{0}$ such that

$$\theta \cdot a \geq \theta \cdot x \text{ for all } a \in \text{cl}(A).$$

The idea of our proof will be

1. Let $\bar{e} \equiv (e_1, \ldots, e_I)$ be the total endowment available in the economy
2. Consider the set A of total endowments that could be distributed such that every agent is strictly better off than e
3. A is convex and $\bar{e} \notin A$; consider the θ that "separates" A from $\bar{e}$
4. Show that given prices $p = \theta$, we have a WE at p and e

Proving the Second Welfare Theorem I

Proof.

Let $A^i \in \mathbb{R}^L_+$ be the set of allocations to i strictly preferred to e^i (i.e., the strict upper contour set of e^i):

$$A^i \equiv \{a^i \in \mathbb{R}^L_+ : u^i(a^i) > u^i(e^i)\}.$$

Concavity of $u^i(\cdot)$ implies quasiconcavity, so A^i is convex.

Consider the set $A \in \mathbb{R}^L_+$ of asset vectors that could be distributed in such a way that every agent is strictly better off than at e:

$$A \equiv \sum_{i \in \mathcal{I}} A^i \equiv \left\{a \in \mathbb{R}^L_+ : \exists a^1 \in A^1, \dots, a^I \in A^I \text{ with } a = \sum_{i \in \mathcal{I}} a^i \right\}.$$

A is also a convex set.

Proving the Second Welfare Theorem II

Proof (continued).

Let $\bar{e} \equiv \sum_i e^i$ be the total amount of assets available in the economy. We know $\bar{e} \notin A$, since then there would be some distribution of $\bar{e}$ that makes every agent strictly better off than e (so e could not be PO).

By increasing preferences, if $a \gg \bar{e}$ we must have $a \in A$.

By the SHT, there is $\theta \neq \mathbf{0}$ such that $\theta \cdot a \geq \theta \cdot \bar{e}$ for all $a \in \text{cl}(A)$. Suppose for some ℓ we had $\theta_\ell < 0$.

- Consider $a \equiv (\bar{e}_1 + \varepsilon, \ldots, \bar{e}_{\ell-1}, \infty, \bar{e}_{\ell+1}, \ldots, \bar{e}_L + \varepsilon)$
- $a \gg \bar{e}$ so $a \in A$
- $\theta \cdot a = -\infty$, so we can't have $\theta \cdot a \geq \theta \cdot \bar{e}$

Thus $\theta > \mathbf{0}$.

Proving the Second Welfare Theorem III

Proof (continued).

We seek to show that $p = \theta$ and e are a WE; this means that markets clear (which they obviously do), and that each agent i maximizes utility at e^i given these prices.
Consider any $a^i \in A^i$ (i.e., any allocation that i strictly prefers to e^i); we must show it is unaffordable at $p = \theta$.

By continuity of preferences, if $u^i(a^i) > u^i(e^i)$, then for λ just below 1, we have $u^i(\lambda a^i) > u^i(e^i)$. Thus by our SHT result,

$$\theta \cdot (\bar{e} - e^i + \lambda a^i) \geq \theta \cdot \bar{e} \implies \theta \cdot \lambda a^i \geq \theta \cdot e^i.$$

Since $p = \theta \neq \mathbf{0}$, $e^i \gg \mathbf{0}$, and $\lambda < 1$, we have $p \cdot a^i > p \cdot e^i$. That is, a^i is unaffordable. □

The welfare theorems

Theorem (First Welfare Theorem)

Suppose $u^i(\cdot)$ is increasing for all $i \in \mathcal{I}$.

If p and $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium, then the allocations $(x^i)_{i \in \mathcal{I}}$ are Pareto optimal.

Theorem (Second Welfare Theorem)

Suppose $u^i(\cdot)$ is continuous, increasing, and concave for all $i \in \mathcal{I}$. Further suppose $e^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$.

If $(e^i)_{i \in \mathcal{I}}$ are Pareto optimal, then there exist prices $p \in \mathbb{R}^l_+$ such that p and $(e^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium.

Thoughts on the second welfare theorem

- The assumptions behind the SWT are much stronger than the FWT—in particular the requirement of convex preferences
- $e^i \gg \mathbf{0}$ is required to ensure that each agent has a positive endowment of some good with a non-zero price—that is, everyone has non-zero wealth
- The SWT is often attributed more importance than it deserves; it says what it says: PO allocations can be supported as WE by some prices

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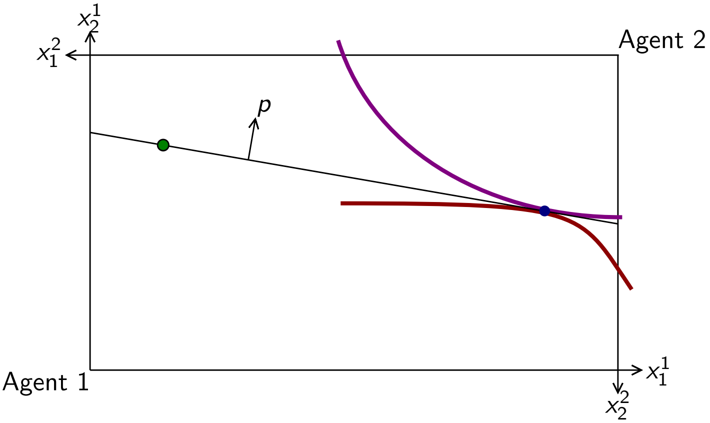
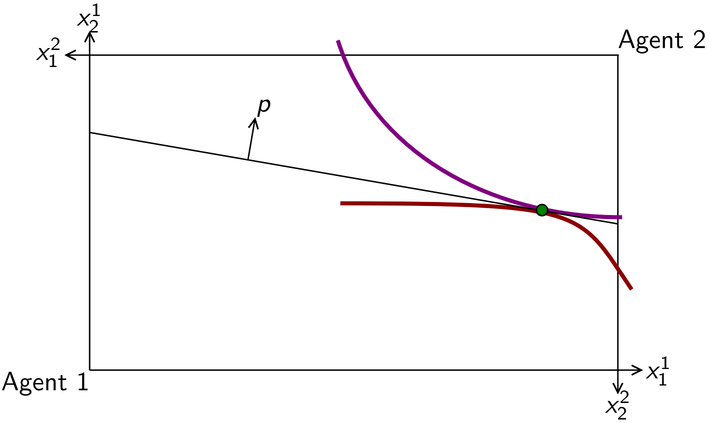
Walrasian equilibria are Pareto optimal. . .



. . . and vice versa (sort of)



Assumptions allowing us to rely on Kuhn-Tucker

Suppose for all $i \in \mathcal{I}$,

1. $u^i(\cdot)$ is continuous;
2. $u^i(\cdot)$ is strictly increasing; i.e., $u^i(x^{i\prime}) > u^i(x^i)$ for any $x^{i\prime} > x^i$;
3. $u^i(\cdot)$ is concave;
4. $u^i(\cdot)$ is differentiable; and
5. $e^i \gg \mathbf{0}$; i.e., every agent has at least a little bit of every good.

Finding Pareto optimal allocations

Consider the following algorithm for finding a Pareto optimal allocation:

1. Decide how much utility to give to each agent $i \in \{2, \ldots, I\}$
2. Maximize agent 1's utility subject to our decision in step 1

That is,

$$\max_{x \in \mathbb{R}_+^{IL}} u^1(x^1)$$

such that

$$\begin{aligned} u^i(x^i) \geq \bar{u}^i \quad & \text{for } i = 2, \ldots, I \\ \textstyle\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i \quad & \text{for } \ell = 1, \ldots, L \end{aligned}$$

Solving the Pareto problem

A Pareto problem

Maximize $u^1(x^1)$ such that

$$\begin{aligned} x &\geq \mathbf{0}; & \\ u^i(x^i) &\geq \bar{u}^i & \text{for } i = 2, \dots, I; \\ \sum_{i \in \mathcal{I}} x^i_\ell &\leq \sum_{i \in \mathcal{I}} e^i_\ell & \text{for } \ell = 1, \dots, L. \end{aligned}$$

Under our assumptions, all the promise-keeping and feasibility constraints must be binding, thus multipliers > 0:

- λ^i multiplier on $u^i(x^i) \geq \bar{u}^i$
- μ_ℓ multiplier on $\sum_i x^i_\ell \leq \sum_i e^i_\ell$

Applying Kuhn-Tucker to the Pareto problem I

$$\mathcal{L} = u^1(x^1) + \sum_{i=2}^{I} \lambda^i \left[u^i(x^i) - \bar{u}^i \right] + \sum_{\ell=1}^{L} \sum_{i=1}^{I} \left[\mu_\ell (e^i_\ell - x^i_\ell) + \gamma^i_\ell x^i_\ell \right]$$

Gives (summarized) FOC

$$\lambda^i \frac{\partial u^i}{\partial x^i_\ell} \leq \mu_\ell \text{ with equality if } x^i_\ell > 0$$

where $\lambda^1 \equiv 1$

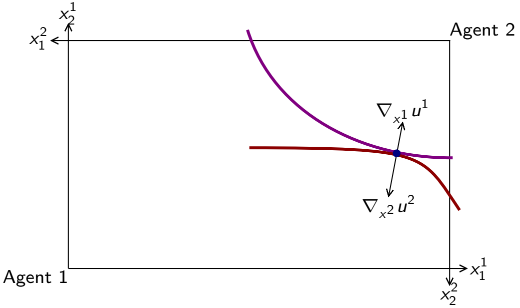
We can interpret the condition by noting that each Lagrange multiplier is the "shadow price" associated with its constraint:

- λ^i is agent 1's extra utility if we take a util away from agent i
- μ_ℓ is agent 1's extra utility gets if we have an extra unit of good ℓ

Applying Kuhn-Tucker to the Pareto problem II

Assuming $x \gg 0$, the FOC is $\lambda^i \frac{\partial u^i}{\partial x^i_\ell} = \mu_\ell$, hence

$$\text{MRS}^i_{k\ell} \equiv \frac{\frac{\partial u^i}{\partial x^i_k}}{\frac{\partial u^i}{\partial x^i_\ell}} = \frac{\mu_k}{\mu_\ell} = \frac{\frac{\partial u^j}{\partial x^j_k}}{\frac{\partial u^j}{\partial x^j_\ell}} \equiv \text{MRS}^j_{k\ell}$$



Maximizing a Bergson-Samuelson social welfare function

Consider a planner who simply maximizes a weighted average of individual utilities:

$$\max_{x \in \mathbb{R}_+^{IL}} \sum_{i \in \mathcal{I}} \beta^i u^i(x^i)$$

such that $\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i$ for $\ell = 1, \dots, L$

$$\mathcal{L} = \sum_{i \in \mathcal{I}} \beta^i u^i(x^i) + \sum_{\ell=1}^{L} \sum_{i=1}^{I} \left[\mu_\ell (e_\ell^i - x_\ell^i) + \gamma_\ell^i x_\ell^i \right]$$

Gives (summarized) FOC

$$\beta^i \frac{\partial u^i}{\partial x_\ell^i} \leq \mu_\ell \text{ with equality if } x_\ell^i > 0$$

Pareto optimality and Bergson-Samuelson SWFs

Pareto problem

- $x^i \geq \mathbf{0}$
- $u^i(x^i) \geq \bar{u}^i$
- $\sum_i x^i_\ell \leq \sum_i e^i_\ell$
- $\lambda^i \frac{\partial u^i}{\partial x^i_\ell} \leq \mu_\ell$ with equality if $x^i_\ell > 0$

Bergson-Samuelson problem

- $x^i \geq \mathbf{0}$
- $\sum_i x^i_\ell \leq \sum_i e^i_\ell$
- $\beta^i \frac{\partial u^i}{\partial x^i_\ell} \leq \mu_\ell$ with equality if $x^i_\ell > 0$

A solution to one is a solution to the other when setting

- $\bar{u}^i = u^i(x^i)$
- $\lambda^i = \beta^i$
- $\mu_\ell = \mu_\ell$

The Walrasian problem

A Walrasian problem

Each individual maximizes $\max_{x^i \in \mathbb{R}^L_+} u^i(x^i)$ such that $p \cdot x^i \leq p \cdot e^i$.

Markets clear: $\sum_{i \in \mathcal{I}} x^i_\ell \leq \sum_{i \in \mathcal{I}} e^i_\ell$ for all $\ell \in \mathcal{L}$.

This gives Lagrangians (for the individual problems)

$$\mathcal{L}^i = u^i(x^i) + \nu^i p \cdot (e^i - x^i) + \sum_{\ell=1}^{L} \gamma^i_\ell x^i_\ell$$

and (summarized) FOCs

$$\frac{\partial u^i}{\partial x^i_\ell} \leq \nu^i p_\ell \text{ with equality if } x^i_\ell > 0$$

The first welfare theorem

Pareto problem

- $x^i \geq \mathbf{0}$
- $u^i(x^i) \geq \bar{u}^i$
- $\sum_i x^i_\ell = \sum_i e^i_\ell$
- $\lambda^i \frac{\partial u^i}{\partial x^i_\ell} \leq \mu_\ell$ with equality if $x^i_\ell > 0$

Walrasian problem

- $x^i \geq \mathbf{0}$
- $p \cdot x^i \leq p \cdot e^i$
- $\sum_i x^i_\ell = \sum_i e^i_\ell$
- $\frac{\partial u^i}{\partial x^i_\ell} \leq \nu^i p_\ell$ with equality if $x^i_\ell > 0$

If (x, p) is a Walrasian equilibrium, we can get that x is Pareto optimal by setting

- $\bar{u}^i = u^i(x^i)$
- $\lambda^i = {}^1/_{\nu^i}$
- $\mu_\ell = p_\ell$

The second welfare theorem

Pareto problem

- $x^i \geq \mathbf{0}$
- $u^i(x^i) \geq \bar{u}^i$
- $\sum_i x^i_\ell = \sum_i e^i_\ell$
- $\lambda^i \frac{\partial u^i}{\partial x^i_\ell} \leq \mu_\ell$ with equality if $x^i_\ell > 0$

Walrasian problem

- $x^i \geq \mathbf{0}$
- $p \cdot x^i \leq p \cdot e^i$
- $\sum_i x^i_\ell = \sum_i e^i_\ell$
- $\frac{\partial u^i}{\partial x^i_\ell} \leq \nu^i p_\ell$ with equality if $x^i_\ell > 0$

If x is Pareto optimal, we can get a Walrasian equilibrium (x, p) by setting

- $e^i = x^i$
- $\nu^i = {}^1/_{\lambda^i}$
- $p_\ell = \mu_\ell$

Part X

General Equilibrium 2

Outline

- Existence of Walrasian equilibria
- Properties of Walrasian equilibria
    - Uniqueness
    - Stability
    - Testable restrictions
- A useful restriction: the "gross substitutes" property
- General equilibrium with production

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Walrasian equilibrium

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium iff

1. All agents maximizing their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$

2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell.$$

Do Walrasian equilibria exist for every economy?

Theorem

Suppose for all $i \in \mathcal{I}$,

1. $u^i(\cdot)$ *is continuous;*
2. $u^i(\cdot)$ *is increasing; i.e.,* $u^i(x') > u^i(x)$ *for any* $x' \gg x$*;*
3. $u^i(\cdot)$ *is concave; and*
4. $e^i \gg \mathbf{0}$*; i.e., every agent has at least a little bit of every good.*

There exist prices $p \in \mathbb{R}^l_+$ *and allocations* $(x^i)_{i \in \mathcal{I}}$ *such that* p *and* x *are a Walrasian equilibrium.*

Excess demand

Definition (excess demand)

The excess demand of agent i is

$$z^i(p) \equiv x^i(p, p \cdot e^i) - e^i,$$

where $x^i(p, w)$ is i's Walrasian demand correspondence.

Aggregate excess demand is

$$z(p) \equiv \sum_{i \in \mathcal{I}} z^i(p).$$

If $p \in \mathbb{R}^L_+$ satisfies $z(p) = \mathbf{0}$, then p and $\left(x^i(p, p \cdot e^i)\right)_{i \in \mathcal{I}}$ are a Walrasian equilibrium

A few notes on excess demand I

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

Under the assumptions of our existence theorem ($u^i(\cdot)$ is continuous, increasing, and concave, and $e^i \gg \mathbf{0}$ for all i):

- $z(\cdot)$ is continuous
  - Continuity of u^i implies continuity of x^i

A few notes on excess demand II

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

- $z(\cdot)$ is homogeneous of degree zero
  - $x^i(p, w^i)$ is homogeneous of degree zero
  - $x^i(p, p \cdot e^i)$ is homogeneous of degree zero in p
  - $z^i(p) \equiv x^i(p, p \cdot e^i) - e^i$ is homogeneous of degree zero
  - $z(p) \equiv \sum_i z^i(p)$ is homogeneous of degree zero

This implies we can normalize one price

A few notes on excess demand III

$$z(p) \equiv \sum_{i \in \mathcal{I}} x^i(p, p \cdot e^i) - \sum_{i \in \mathcal{I}} e^i$$

- $p \cdot z(p) = 0$ for all p (Walras' Law for excess demand)
  - By Walras' Law, $p \cdot x^i(p, w^i) = w^i$
  - $p \cdot x^i(p, p \cdot e^i) = p \cdot e^i$
  - $p \cdot z^i(p) \equiv p \cdot \big(x^i(p, p \cdot e^i) - e^i\big) = 0$
  - $p \cdot z(p) \equiv p \cdot \sum_i z^i(p) = 0$

Suppose all but one market clear; i.e., $z_2(p) = \cdots = z_L(p) = 0$

$$p \cdot z(p) = p_1 z_1(p) + \underbrace{p_2 z_2(p) + \cdots + p_L z_L(p)}_{=0} = 0$$

by Walras' Law; hence $z_1(p) = 0$ as long as $p_1 > 0$

Thus if all but one market clear, the final market must also clear

W.E. requires a solution to $z(p) = 0$ I

Consider a two-good economy

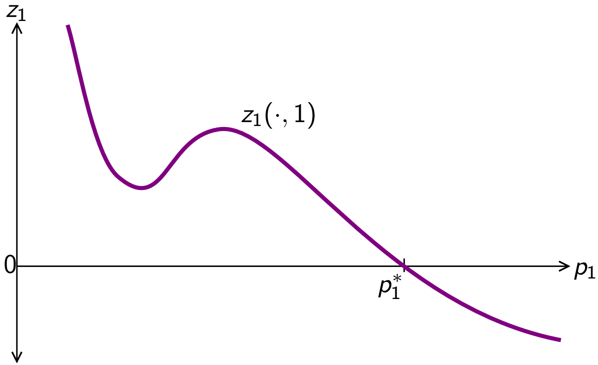
- Normalize $p_2 = 1$ by homogeneity of degree zero of $z(\cdot)$
- As long as the good one market clears, the good two market will as well (by Walras' Law)

We can find a W.E. whenever $z_1(p_1, 1) = 0$

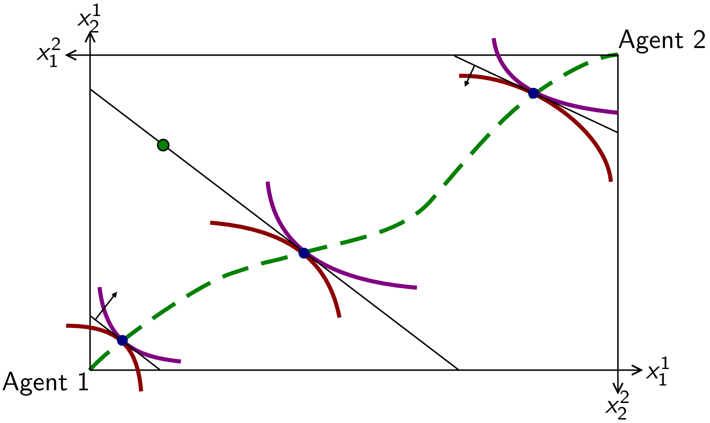
- $z_1(\cdot, 1)$ is continuous
- As $p_1 \to 0$, excess demand for good one must go to infinity since preferences are increasing and $e_2^i > 0$ for all i
- As $p_1 \to \infty$, excess demand for good one must be negative since preferences are increasing and $e_1^i > 0$ for all i

W.E. requires a solution to $z(p) = 0$ II

By an intermediate value theorem, there is at least one W.E.



W.E. in the Edgeworth box economy



Outline

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Other properties of Walrasian equilibria

We have established that an economy satisfying certain properties, at least one Walrasian equilibrium exists

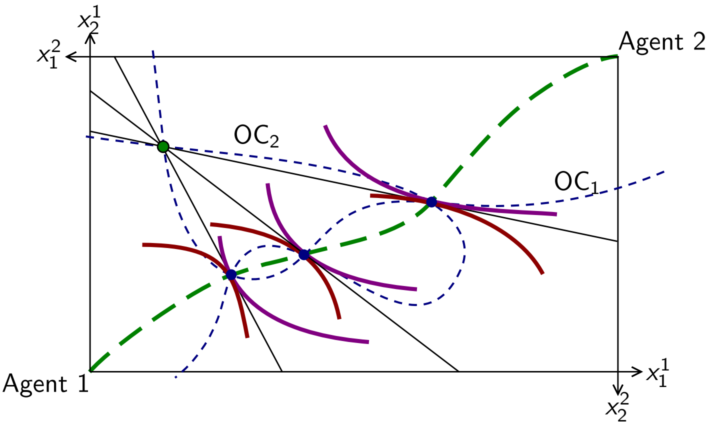
Other questions include:

1. How many Walrasian equilibria are there?
2. How does an economy (as distinct from an economist) “find” equilibrium?
3. Can we test the Walrasian model in the data?

Uniqueness of Walrasian equilibria: Edgeworth box

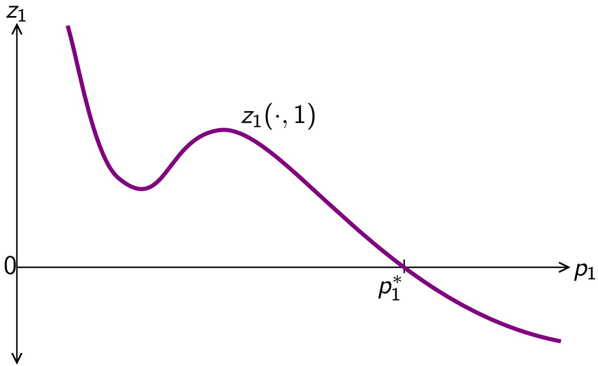
Question 1: Uniqueness

Is there a unique Walrasian equilibrium (up to price normalization)? If not, how many Walrasian equilibria are there?



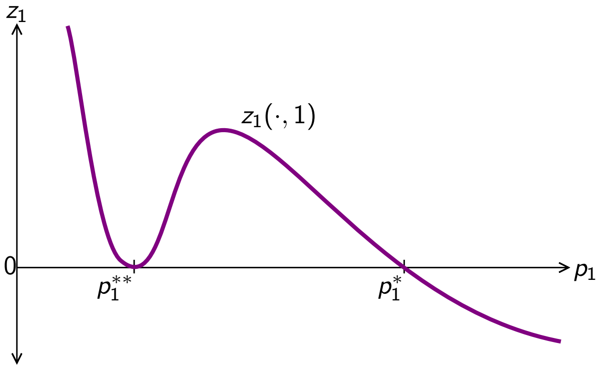
Uniqueness of Walrasian equilibria I

There could be one Walrasian equilibrium



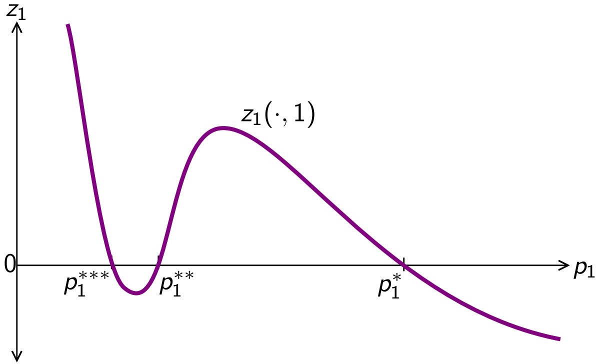
Uniqueness of Walrasian equilibria II

There could be two W.E. (although this is "non-generic")



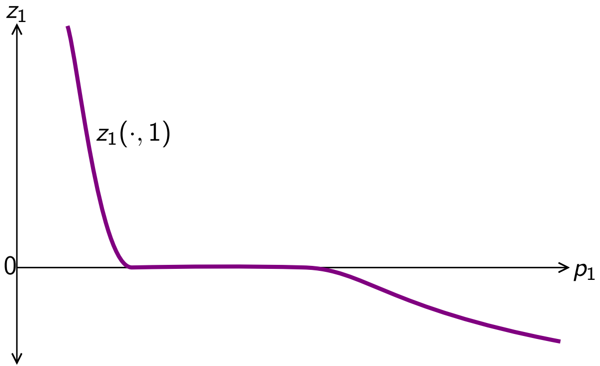
Uniqueness of Walrasian equilibria III

There could be three W.E.



Uniqueness of Walrasian equilibria IV

There could be infinite W.E. (although again, not generically)



Observations on multiplicity of Walrasian equilibria

It seems (and can be rigorously shown) that:

- W.E. are generally not globally unique
- W.E. are locally unique (generically)
- There are a finite number of W.E. (generically)
- There are an odd number of W.E. (generically)

Stability of Walrasian equilibria

Question 2: Stability

Is a Walrasian equilibrium "stable," in the sense that a reasonable dynamic adjustment process converges to equilibrium prices and quantities?

Underlying question is: How does the economy "find" prices?

- Hard to say in real world where prices come from
- Proposed idea: a dynamic adjustment mechanism that converges to W.E. prices

Walrasian tatonnement

One possibility

1. "Walrasian auctioneer" suggests prices
2. Agents report demand at these prices
3. If excess demand is non-zero, return to step 1

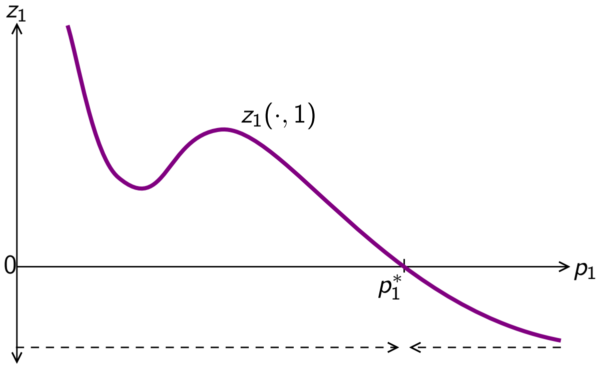
Possible price adjustment rule:

$$p(t+1) = p(t) + \alpha(t)\, z\big(p(t)\big)$$

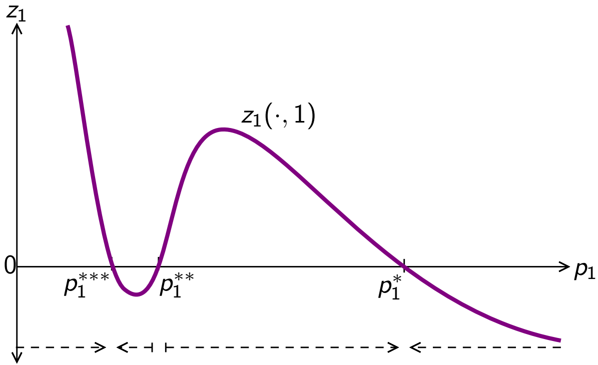
Big problems:

- Unrealistic description of how the economy really works
- No incentives to honestly report demand
- Not necessarily stable

Possible stability of Walrasian tatonnement



Possible instability of Walrasian tatonnement



Testable restrictions implied by the Walrasian model

Question 3: Testability

Does Walrasian equilibrium impose meaningful restrictions on observable data?

We noted several properties of excess demand $z(p)$:

- Continuity
- Homogeneity of degree zero
- Walras' Law ($p \cdot z(p) = 0$ for all p)
- Limit properties

Actually, this is all we get

Anything goes

Theorem (Sonnenschein-Mantel-Debreu)

Consider a continuous function $f : B \to \mathbb{R}^L$ on an open and bounded set $B \subseteq \mathbb{R}^L_{++}$ such that

- *$f(\cdot)$ is homogeneous of degree zero, and*
- *$p \cdot f(p) = 0$ for all $p \in B$.*

Then there exists an economy (goods, agents, preferences, and endowments) with aggregate excess demand function $z(\cdot)$ satisfying $z(p) = f(p)$ for all $p \in B$.

Often interpreted as "anything goes" in terms of comparative statics. . . actually not quite right

If we are prepared to further restrict preferences, can get more robust predictions

Outline

- Existence of Walrasian equilibria
- Properties of Walrasian equilibria
  - Uniqueness
  - Stability
  - Testable restrictions
- A useful restriction: the "gross substitutes" property
- General equilibrium with production

Gross substitutes and the gross substitutes property I

Recall that

Definition (Gross substitute—partial equilibrium)

Good ℓ is a (strict) gross substitute for good m iff $x_\ell(p, w)$ is (strictly) increasing in p_m.

In our G.E. framework, wealth depends on prices ($w = e \cdot p$) so

Definition (Gross substitute—general equilibrium)

Good ℓ is a (strict) gross substitute for good m iff $x_\ell(p, e \cdot p)$ is (strictly) increasing in p_m.

Gross substitutes and the gross substitutes property II

Definition (Gross substitutes property)

Marshallian demand function $x(p) \equiv x(p, e \cdot p)$ has the (strict) gross substitutes property if every good is a (strict) gross substitute for every other good.

More generally. . .

Definition (Gross substitutes property)

A function $f(\cdot)$ has the (strict) gross substitutes property if $f_\ell(p)$ is (strictly) increasing in p_m for all $\ell \neq m$.

Gross substitutes and the gross substitutes property III

Suppose each individual's Marshallian demand satisfies the gross substitutes property; i.e., $x^i_\ell(p)$ is increasing in p_m for all $\ell \neq m$

Then

- Individual excess demands also satisfy it: $z^i_\ell(p) \equiv x^i_\ell(p) - e^i_\ell$ is increasing in p_m
- Aggregate excess demand also satisfies it: $z_\ell(p) \equiv \sum_i z^i_\ell(p)$ is increasing in p_m

Uniqueness of Walrasian equilibrium I

Theorem

If aggregate excess demand $z(\cdot)$ satisfies the strict gross substitutes property, then the economy has at most one Walrasian equilibrium (up to price normalization).

Proof.

Suppose in contradiction that there are two non-collinear Walrasian equilibrium prices p and p'; i.e., $z(p) = z(p') = \mathbf{0}$. Define $\lambda_\ell \equiv p'_\ell / p_\ell$, and consider $\tilde{\ell} \equiv \operatorname{argmax}_\ell \lambda_\ell$. Finally, define $\tilde{p} \equiv \lambda_{\tilde{\ell}} p$. This normalization ensures that $\tilde{p}_{\tilde{\ell}} = p'_{\tilde{\ell}}$, and

$$\tilde{p}_\ell = \lambda_{\tilde{\ell}} p_\ell \geq \lambda_\ell p_\ell = p'_\ell,$$

with strict inequality for some ℓ (since otherwise $p' = \lambda_{\tilde{\ell}} p$).

Uniqueness of Walrasian equilibrium II

Proof (continued).

Consider moving from p' to $\tilde{p}$ by increasing the price of each good one at a time. By gross substitutes,

$$\begin{aligned} 0 = z_{\tilde{\ell}}(p') &\leq z_{\tilde{\ell}}(\tilde{p}_1, p'_2, \ldots, p'_L) \\ &\leq z_{\tilde{\ell}}(\tilde{p}_1, \tilde{p}_2, p'_3, \ldots, p'_L) \\ &\vdots \\ &< z_{\tilde{\ell}}(\tilde{p}) \end{aligned}$$

where strict inequality obtains since $\tilde{p}_\ell > p'_\ell$ for some ℓ.

By homogeneity of degree zero of $z(\cdot)$, we have $z_{\tilde{\ell}}(\tilde{p}) = z_{\tilde{\ell}}(\lambda_{\tilde{\ell}} p) = z_{\tilde{\ell}}(p) = 0$, a contradiction. □

Other implications of gross substitutes

The gross substitutes property can be used to show a number of other properties of Walrasian equilibrium; e.g.,

- Walrasian tatonnement will converge to the unique equilibrium
- Any change that raises the excess demand of a good will increase its equilibrium price

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- Properties of Walrasian equilibria
  - Uniqueness
  - Stability
  - Testable restrictions
- A useful restriction: the "gross substitutes" property
- General equilibrium with production

Adding production to the Walrasian model

So far our exchange economy has treated the stock of goods available as fixed through endowments

Now add K firms $k \in \mathcal{K} \equiv \{1, \ldots, K\}$

- Firm k has production set $Y^k \subseteq \mathbb{R}^L$
- Will make a number of "standard" producer theory assumptions
- Also need some additional assumptions to make sure economy is well behaved

Final primitive: what happens to firms' profits? We typically assume firms are owned by consumers

Additional assumptions on production

None of the following assumptions should surprise at this point, but should be kept in mind when interpreting our following results:

- Firms are price takers (as are consumers)
- Technology is exogenously given
- Firms maximize profits

The Walrasian Model of the production economy I

Primitives of the model

- L goods $\ell \in \mathcal{L} \equiv \{1, \ldots, L\}$
- I consumers $i \in \mathcal{I} \equiv \{1, \ldots, I\}$
  - Endowments $e^i \in \mathbb{R}^L_+$; consumers do not have monetary wealth, but rather an endowment of goods which they can trade or consume
  - Preferences represented by utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$
- K firms $k \in \mathcal{K} \equiv \{1, \ldots, K\}$
  - Production sets $Y^k \subseteq \mathbb{R}^L$
  - Ownership structure $(\alpha^{ki})_{k \in \mathcal{K}, i \in \mathcal{I}}$ where α^{ki} is consumer i's share of firm k
- Endogenous prices $p \in \mathbb{R}^L_+$, taken as given by each consumer and firm

The Walrasian Model of the production economy II

Each consumer i solves

$$\max_{x \in B^i(p)} u^i(x)$$

where

$$B^i(p) \equiv \left\{ x \in \mathbb{R}^L_+ : p \cdot x \leq p \cdot e^i + \sum_{k \in K} \alpha^{ki} (p \cdot y^k) \right\}$$

Each firm k solves

$$\max_{y^k \in Y^k} p \cdot y^k$$

Walrasian equilibrium

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i \in \mathcal{I}}$ and $(y^k)_{k \in \mathcal{K}}$ are a Walrasian equilibrium iff

1. All consumers maximize their utilities; i.e., for all $i \in \mathcal{I}$,
$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$
2. All firms maximize their profits; i.e., for all $k \in \mathcal{K}$,
$$y^k \in \operatorname*{argmax}_{y \in Y^k} p \cdot y;$$
3. Markets clear; i.e., for all $\ell \in \mathcal{L}$,
$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell + \sum_{k \in \mathcal{K}} y^k_\ell.$$

Pareto optimality

Definition (feasible alocation)

Allocations $(x^i)_{i\in\mathcal{I}} \in \mathbb{R}_+^{I\cdot L}$ and production plan $(y^k)_{k\in\mathcal{K}} \in \mathbb{R}^{I\cdot L}$ are feasible iff $y^k \in Y^k$ for all $k \in \mathcal{K}$, and for all $\ell \in \mathcal{L}$,

$$\sum_{i\in\mathcal{I}} x_\ell^i \leq \sum_{i\in\mathcal{I}} e_\ell^i + \sum_{k\in\mathcal{K}} y_\ell^k.$$

Definition (Pareto optimality)

Allocations $(x^i)_{i\in\mathcal{I}}$ and production plan $(y^k)_{k\in\mathcal{K}}$ are Pareto optimal iff

1. x and y are feasible, and
2. There are no other feasible allocations $\hat{x}$ and $\hat{y}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

The First Welfare Theorem

Theorem (First Welfare Theorem)

Suppose $u^i(\cdot)$ is increasing (i.e., $u^i(x') > u^i(x)$ for any $x' \gg x$) for all $i \in \mathcal{I}$.

If p and $(x^i)_{i\in\mathcal{I}}$ and $(y^k)_{k\in\mathcal{K}}$ are a Walrasian equilibrium, then the allocations $(x^i)_{i\in\mathcal{I}}$ and $(y^k)_{k\in\mathcal{K}}$ are Pareto optimal.

The Second Welfare Theorem

Theorem (Second Welfare Theorem)

Suppose for all $i \in \mathcal{I}$,

1. $u^i(\cdot)$ *is increasing; i.e.,* $u^i(x') > u^i(x)$ *for any* $x' \gg x$*;*
2. $u^i(\cdot)$ *is concave; and*
3. $e^i \gg \mathbf{0}$*; i.e., every agent has at least a little bit of every good.*

Further suppose that production sets Y^k *are closed and convex for all* $k \in K$*, which rules out increasing returns to scale.*

Suppose $(x^i)_{i \in \mathcal{I}}$ *and* $(y^k)_{k \in \mathcal{K}}$ *are Pareto optimal, and that* $x^i \gg \mathbf{0}$ *for all* $i \in \mathcal{I}$*.*

Then there exist prices $p \in \mathbb{R}^l_+$*, ownership shares* $(\alpha^{ki})_{k \in \mathcal{K}, i \in \mathcal{I}}$*, and transferred endowments* $(\tilde{e}^i)_{i \in \mathcal{I}}$ *where* $\sum_i e^i = \sum_i \tilde{e}^i$ *such that* p *and* $(x^i)_{i \in \mathcal{I}}$ *and* $(y^k)_{k \in \mathcal{K}}$ *are a Walrasian equilibrium in the economy where endowments are* $(\tilde{e}^i)_{i \in \mathcal{I}}$*.*

Do Walrasian equilibria exist for every economy?

Theorem

Suppose

- $u^i(\cdot)$ *is continuous, increasing, and concave for all* $i \in \mathcal{I}$;
- $e^i \gg \mathbf{0}$ *for all* $i \in \mathcal{I}$;
- *Production sets* Y^k *are closed and convex, and have shutdown and free disposal for all* $k \in K$;
- $$\left[\sum_{k \in \mathcal{K}} Y^k\right] \cap \left[-\sum_{k \in \mathcal{K}} Y^k\right] = \{\mathbf{0}\},$$
  which rules out the possibility that firms can cooperate to produce unlimited output.

Then there exists a Walrasian equilibrium.

Firms with constant returns to scale technology

Suppose a firm has CRS production technology; i.e.,
$y \in Y \implies \beta y \in Y$ for all y and all $\beta > 0$

What can we say about its profit?

- Can it be strictly positive? No...otherwise it could scale up production arbitrarily and achieve infinite profit
- Could be zero due to prices
- Could be zero due to shutdown

So ownership structure $(\alpha^{ki})_{ki}$ doesn't matter

Part XI

Wrap-up Summary

What have we done this quarter? We have...

- Discussed a number of assumptions
- Learned to work with optimization problems
- Described how and when to represent preferences with utility functions
- Discussed comparative statics properties of demand
- Measured changes in consumer welfare, created price indices, and defined optimality
- Measured the risk of uncertain prospects
- Defined Walrasian equilibrium, and discussed its properties

Let's be honest

graduate school: it's like looking directly into the bulb of a high-powered flashlight for two years, only more expensive

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Role of simplifying assumptions

No consensus about "correct" view

Modeling is an abstraction

- Relies on simplifying but untrue assumptions
- Highlight important effects by suppressing other effects
- Basis for numerical calculations

Models can be useful in different ways

- Relevant predictions reasonably accurate; can sometimes be checked using data or theoretical analysis
- Failure of relevant predictions can highlight which simplifying assumptions are most relevant
- "Usual" or "standard" models often fail realism checks; do not skip validation

Choice theory: simplifying assumptions

Simplifying assumptions include:

- Choices are made from some feasible set
- Preferred things get chosen
- Any pair of potential choices can be compared
- Preferences are transitive

In the case of uncertainty:

- Finite number of outcomes, or else outcomes in $\mathbb{R}$
- Objectively known probability distributions over outcomes
- Complete and transitive preferences over lotteries
- Continuous and “independent” preferences over lotteries

Producer theory: simplifying assumptions

Simplifying assumptions include:

- Firms are price takers (both input and output markets)
- Technology is exogenously given
- Firms maximize profits; should be true as long as
    - The firm is competitive
    - There is no uncertainty about profits
    - Managers are perfectly controlled by owners

Consumer theory: simplifying assumptions

Simplifying assumptions include:

- Utility function is general, but assumed to exist
- Choice set defined by linear budget constraint
  - Consumers are price takers
  - Prices are linear
  - Perfect information: prices are all known
- Finite number of goods
  - Goods are described by quantity and price
  - Goods are divisible
  - Goods may be time- or situation-dependent
  - Perfect information: goods are all well understood

General equilibrium: simplifying assumptions

Simplifying assumptions include:

- All agents face the same prices
- Markets exist for all goods
- Agents can freely participate in markets without cost
- "Standard" consumer theory assumptions
    - Preferences can be represented by a utility function
    - All agents are price takers
    - Finite number of divisible goods
    - Linear prices
    - Perfect information about goods and prices

In the case of G.E. with production:

- Firms are price takers (as are consumers)
- Technology is exogenously given
- Firms maximize profits

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Tools to find optimization objects

- Choice correspondence
  - Find using Kuhn-Tucker
  - Find using Envelope Theorem (often)
  - Comparative statics using Topkis' Theorem
- Value function
  - Find using "adding-up"
  - Comparative statics using Envelope Theorem
- Feasible set
  - Describe using inner and outer bounds

We can simplify a number of problems

1. When we approach the same economic problem different ways, these are the same problem and have the same solution
    - Profit maximization and cost minimization
    - Utility maximization and expenditure minimization
2. Optimization problems that look different may turn out to be the same problem (e.g., Pareto, Bergson-Samuelson, Walrasian equilibrium)
3. Features of the problem may allow us to turn inequality constraints into equality constraints (e.g., Walras' Law)

Envelope Theorem

Theorem (Envelope Theorem)

Consider a constrained optimization problem $v(\theta) = \max_x f(x, \theta)$ *such that* $g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0$.

Comparative statics on the value function are given by:

$$\frac{\partial v}{\partial \theta_i} = \left.\frac{\partial f}{\partial \theta_i}\right|_{x^*} + \sum_{k=1}^{K} \lambda_k \left.\frac{\partial g_k}{\partial \theta_i}\right|_{x^*} = \left.\frac{\partial \mathcal{L}}{\partial \theta_i}\right|_{x^*}$$

(for Lagrangian $\mathcal{L}(x, \theta, \lambda) \equiv f(x, \theta) + \sum_k \lambda_k g_k(x, \theta)$*) for all* θ *such that the set of binding constraints does not change in an open neighborhood.*

The derivative of the value function equals the derivative of the Lagrangian

Topkis' Theorem

Theorem (Topkis' Theorem)

Suppose

1. $F : X \times T \to \mathbb{R}$ *(for X a lattice, T partially ordered)*
   - *is supermodular in x (i.e., ID in all (x_i, x_j))*
   - *has ID in (x, t) (i.e., ID in all (x_i, t_j))*
2. $t' > t$,
3. $x \in X^*(t) \equiv \operatorname{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then

$$(x \wedge x') \in X^*(t) \textit{ and } (x \vee x') \in X^*(t').$$

That is, $X^*(\cdot)$ is nondecreasing in t in the stronger set order.

If $X^*(\cdot)$ is a function, nondecreasing in the stronger set order reduces to simple nondecreasing

The Kuhn-Tucker algorithm I

The Kuhn-Tucker Theorem provide the key generalization of the "Lagrangian" method for constrained optimization

Consider the problem

$$v(\theta) = \max_{x \in \mathbb{R}^n} f(x, \theta)$$

subject to constraints

$$g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0.$$

Set up a Lagrangian

$$\mathcal{L}(x, \theta, \lambda) \equiv f(x, \theta) + \sum_{k=1}^{K} \lambda_k g_k(x, \theta)$$

The Kuhn-Tucker algorithm II

Theorem (Kuhn-Tucker)

Suppose x^* solves the optimization problem at parameter θ, and

- $f(\cdot, \theta)$ *and* $g_1(\cdot, \theta), \ldots, g_K(\cdot, \theta)$ *are all differentiable in* x*;*
- *the constraint set is non-empty; and*
- *constraint qualification holds.*

Then there exist nonnegative $\lambda_1, \ldots, \lambda_K$ *such that*

1. *first-order conditions hold:*

$$D_x f(x^*, \theta) + \sum_{k=1}^{K} \lambda_k D_x g_k(x^*, \theta) = D_x \mathcal{L}(x^*, \lambda, \theta) = 0;$$

2. $\lambda_k g_k(x^*, \theta) = 0$ *(complementary slackness); and*
3. $g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0$ *(original constraints).*

The Kuhn-Tucker algorithm III

Kuhn-Tucker conditions are necessary and sufficient for a solution (assuming differentiability) as long as we have a "convex problem":

1. The constraint set is convex
   - If each constraint gives a convex set, the intersection is a convex set
   - The set $\{x \colon g_k(x, \theta) \geq 0\}$ is convex as long as $g_k(\cdot, \theta)$ is a quasiconcave function of x
2. The objective function is concave
   - If we only know the objective is quasiconcave, there are other conditions that ensure Kuhn-Tucker is sufficient

"Inner bound" and "outer bound" approaches

Inner bound approach If it's observed, it must be feasible

Outer bound approach If it's (strictly) better than optimal, it must be (strictly) unaffordable; The "outer bound approach" is also known as revealed preference

Generally, rationalizability requires (differentiable case)

Given a linear objective function, generally
Choice function and value function require. . .

1. Adding-up
2. Envelope
3. Convexity/concavity of value function

Choice function requires. . .

1. Homogeneity of degree zero
2. Symmetric positive/negative semidefinite Jacobian

Value function requires. . .

1. Homogeneity of degree one
2. Convexity/concavity

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Caveats of utility representation

- Not all preferences can be represented by a utility function
  - Completeness, transitivity, and either continuity or a finite choice set are sufficient
  - *Also* need continuity and independence for expected utility representation
- (Generally) cannot make interpersonal comparisons
  - Representation robust to increasing monotone transformations
  - *Expected* utility representation only robust to increasing affine transformations

Properties of preferences and utility representations

Property of $\succsim$		Property of $u(\cdot)$
Monotone	$\iff$	Nondecreasing
Strictly monotone	$\iff$	Increasing
Locally non-satiated	$\iff$	Has no local maxima in X
Convex	$\iff$	Quasiconcave
Homothetic	$\Longleftarrow$	Homogeneous of degree one
Separable	$\Longleftarrow$	$U(v(x), y)$
Numeraire	$\Longleftarrow$	Quasilinear
Expected utility	$\iff$	vN-M
Risk-averse exp. util.	$\iff$	vN-M with concave Bernoulli

Several ways to measure attitudes towards risk

Theorem

The following definitions of u being "more risk-averse" than v are equivalent:

1. *Whenever u prefers F to a certain payout d, then v does as well; i.e., for all F and d,*

$$\mathbb{E}_F\left[u(x)\right] \geq u(d) \implies \mathbb{E}_F\left[v(x)\right] \geq v(d);$$

2. *Certain equivalents $c_u(F) \leq c_v(F)$ for all F;*
3. *$u(\cdot)$ is "more concave" than $v(\cdot)$; i.e., there exists some increasing concave function $g(\cdot)$ such that $u(x) = g\left(v(x)\right)$ for all x;*
4. *Arrow-Pratt coefficients of absolute risk aversion $A_u(x) \geq A_v(x)$ for all x.*

Rational choice theory often fails experimentally

Choices appear to be highly situational, depending on

- Other available options
- Way that options are "framed"
- Social context/emotional state

Rational choice depends on a *considered* comparison of options

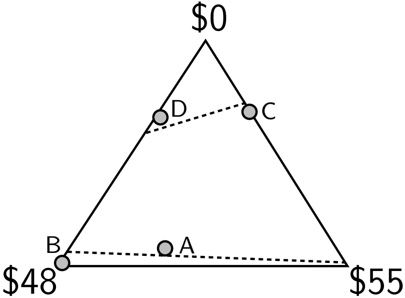
- Pairwise comparison
- Utility maximization

Many actual choices appear to be made using

- Intuitive reasoning
- Heuristics
- Instinctive desire

The vN-M framework often fails experimentally

- The Independence Axiom fails



- Framing matters

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Marshallian response to changes in wealth

Definition (Normal good)

Good i is a normal good if $x_i(p, w)$ is increasing in w.

Definition (Inferior good)

Good i is an inferior good if $x_i(p, w)$ is decreasing in w.

Marshallian response to changes in own price

Definition (Regular good)

Good i is a regular good if $x_i(p, w)$ is decreasing in p_i.

Definition (Giffen good)

Good i is an Giffen good if $x_i(p, w)$ is increasing in p_i.

By the Slutsky equation (which gives $\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial w} x_i$ for $i = j$)

- Normal $\implies$ regular
- Giffen $\implies$ inferior

Marshallian response to changes in other goods' price

Definition (Gross substitute)

Good i is a gross substitute for good j if $x_i(p, w)$ is increasing in p_j.

Definition (Gross complement)

Good i is a gross complement for good j if $x_i(p, w)$ is decreasing in p_j.

Gross substitutability/complementarity is not necessarily symmetric

Hicksian response to changes in other goods' price

Definition (Substitute)

Good i is a substitute for good j if $h_i(p, \bar{u})$ is increasing in p_j.

Definition (Complement)

Good i is a complement for good j if $h_i(p, \bar{u})$ is decreasing in p_j.

Substitutability/complementarity is symmetric

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Quantifying consumer welfare

"How much money is required to achieve a fixed level of utility before and after the price change?"

$$\text{Variation} = e(p, u_{\text{reference}}) - e(p', u_{\text{reference}})$$

1. How much would consumer be willing to pay for the price change?
   Reference: Old utility ($u_{\text{reference}} = \bar{u} \equiv v(p, w)$)
2. How much would we have to pay consumer to miss out on price change?
   Reference: New utility ($u_{\text{reference}} = \bar{u}' \equiv v(p', w)$)

Compensating and equivalent variation

Definition (Compensating variation)

The amount less wealth (i.e., the fee) a consumer needs to achieve the same maximum utility at new prices (p') as she had before the price change (at prices p):

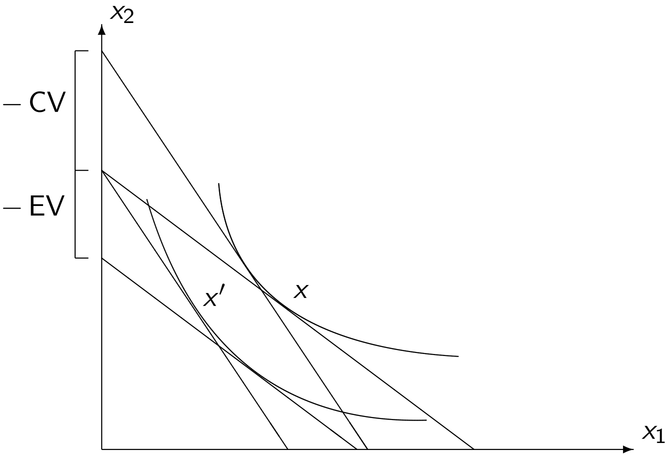
$$\text{CV} \equiv e\big(p, v(p, w)\big) - e\big(p', v(p, w)\big) = w - e\big(p', \underbrace{v(p, w)}_{\equiv \bar{u}}\big).$$

Definition (Equivalent variation)

The amount more wealth (i.e., the bonus) a consumer needs to achieve the same maximum utility at old prices (p) as she could achieve after a price change (to p'):

$$\text{EV} \equiv e\big(p, v(p', w)\big) - e\big(p', v(p', w)\big) = e\big(p, \underbrace{v(p', w)}_{\equiv \bar{u}'}\big) - w.$$

Graphically illustrating CV and EV



Price indices

Definition (ideal index)

$$\text{Ideal Index}(\bar{u}) \equiv \frac{p'_{\text{util}}}{p_{\text{util}}} = \frac{e(p', \bar{u})/\bar{u}}{e(p, \bar{u})/\bar{u}} = \frac{e(p', \bar{u})}{e(p, \bar{u})}.$$

Definition (Laspeyres index)

$$\text{Laspeyres Index} \equiv \frac{p' \cdot x}{p \cdot x} = \frac{p' \cdot x}{w} = \frac{p' \cdot x}{e(p, \bar{u})},$$

where $\bar{u} \equiv v(p, w)$.

Definition (Paasche index)

$$\text{Paasche Index} \equiv \frac{p' \cdot x'}{p \cdot x'} = \frac{w'}{p \cdot x'} = \frac{e(p', \bar{u}')}{p \cdot x'},$$

where $\bar{u}' \equiv v(p', w')$.

Bounding the Laspeyres and Paasche indices

Note that since $u(x) = \bar{u}$ and $u(x') = \bar{u}'$, by "revealed preference"

$$p' \cdot x \geq \min_{\xi:\ u(\xi) \geq \bar{u}} p' \cdot \xi = e(p', \bar{u})$$

$$p \cdot x' \geq \min_{\xi:\ u(\xi) \geq \bar{u}'} p \cdot \xi = e(p, \bar{u}')$$

Thus we get that the Laspeyres index overestimates inflation, while the Paasche index underestimates it:

$$\text{Laspeyres} \equiv \frac{p' \cdot x}{e(p, \bar{u})} \geq \frac{e(p', \bar{u})}{e(p, \bar{u})} \equiv \text{Ideal}(\bar{u})$$

$$\text{Paasche Index} \equiv \frac{e(p', \bar{u}')}{p \cdot x'} \leq \frac{e(p', \bar{u}')}{e(p, \bar{u}')} \equiv \text{Ideal}(\bar{u}')$$

Pareto optimality

Definition (feasible allocation)

Allocations $(x^i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{I \cdot L}$ are feasible iff for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x_\ell^i \leq \sum_{i \in \mathcal{I}} e_\ell^i.$$

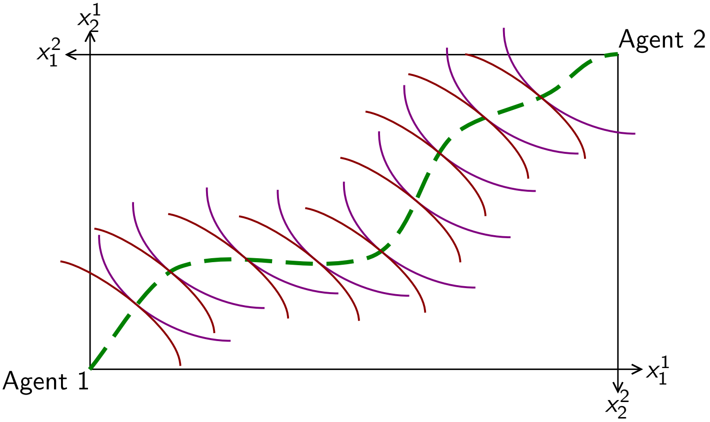
Definition (Pareto optimality)

Allocations $x \equiv (x^i)_{i \in \mathcal{I}}$ are Pareto optimal iff

1. x is feasible, and
2. There is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{I}$ with strict inequality for some i.

The Pareto set

The Pareto set is the locus of Pareto optimal allocations



This quarter, we. . .

- Discussed a number of assumptions
- Learned to work with optimization problems
- Described how and when to represent preferences with utility functions
- Discussed comparative statics properties of demand
- Measured changes in consumer welfare, created price indices, and defined optimality
- Measured the risk of uncertain prospects
- Defined Walrasian equilibrium, and discussed its properties

First-order stochastic dominance

Definition (first-order stochastic dominance)

Distribution G first-order stochastic dominates distribution F iff lottery G is preferred to F under every nondecreasing Bernoulli utility function $u(\cdot)$. That is, for every nondecreasing $u \colon \mathbb{R} \to \mathbb{R}$, the following (equivalent) statements hold:

$$G \succsim_u F,$$

$$\mathbb{E}_G\left[u(x)\right] \geq \mathbb{E}_F\left[u(x)\right],$$

$$\int_{\mathbb{R}} u(x)\, dG(x) \geq \int_{\mathbb{R}} u(x)\, dF(x).$$

Equivalently, G first-order stochastic dominates F iff

- $G(x) \leq F(x)$ for all x.
- We can construct G from F using upward shifts.

Second-order stochastic dominance

Definition (second-order stochastic dominance)

Suppose F and G have the same mean.

Distribution G second-order stochastic dominates distribution F iff lottery G is preferred to F under every concave, nondecreasing Bernoulli utility function $u(\cdot)$. That is, for every concave, nondecreasing $u \colon \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_G\left[u(x)\right] \geq \mathbb{E}_F\left[u(x)\right].$$

Equivalently, G second-order stochastic dominates F iff

- $$\int_{-\infty}^{x} G(t)\,dt \leq \int_{-\infty}^{x} F(t)\,dt \text{ for all } x.$$
- We can construct F from G using mean-preserving spreads.

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Walrasian equilibrium in the exchange economy

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium iff

1. All agents maximize their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$

2. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell.$$

Walrasian equilibrium with production

Definition (Walrasian equilibrium)

Prices p and quantities $(x^i)_{i \in \mathcal{I}}$ and $(y^k)_{k \in \mathcal{K}}$ are a Walrasian equilibrium iff

1. All consumers maximize their utilities; i.e., for all $i \in \mathcal{I}$,

$$x^i \in \operatorname*{argmax}_{x \in B^i(p)} u^i(x);$$

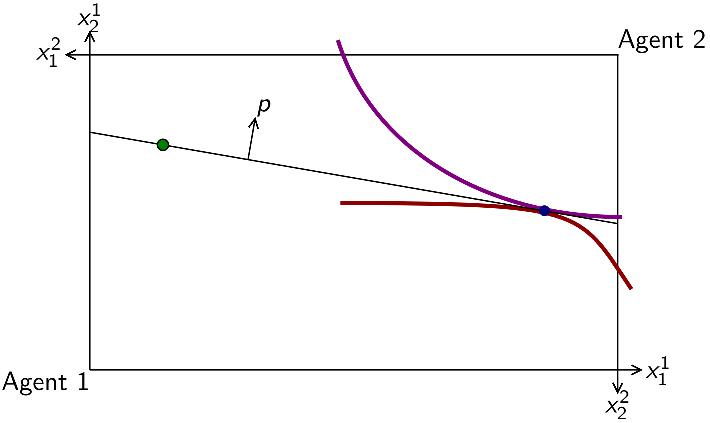
2. All firms maximize their profits; i.e., for all $k \in \mathcal{K}$,

$$y^k \in \operatorname*{argmax}_{y \in Y^k} p \cdot y;$$

3. Markets clear; i.e., for all $\ell \in \mathcal{L}$,

$$\sum_{i \in \mathcal{I}} x^i_\ell = \sum_{i \in \mathcal{I}} e^i_\ell + \sum_{k \in \mathcal{K}} y^k_\ell.$$

Walrasian equilibrium in the Edgeworth box



The welfare theorems

Theorem (First Welfare Theorem)

Suppose $u^i(\cdot)$ is increasing (i.e., $u^i(x') > u^i(x)$ for any $x' \gg x$) for all $i \in \mathcal{I}$.

If p and $(x^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium, then the allocations $(x^i)_{i \in \mathcal{I}}$ are Pareto optimal.

Theorem (Second Welfare Theorem)

Suppose $u^i(\cdot)$ is continuous, increasing, and concave for all $i \in \mathcal{I}$. Further suppose $e^i \gg \mathbf{0}$ for all $i \in \mathcal{I}$.

If $(e^i)_{i \in \mathcal{I}}$ are Pareto optimal, then there exist prices $p \in \mathbb{R}^l_+$ such that p and $(e^i)_{i \in \mathcal{I}}$ are a Walrasian equilibrium.

Properties of Walrasian equilibria

- Under the conditions listed above for the second welfare theorem, a W.E. exists
  - Intermediate value theorem on $z(p) = 0$
  - Intermediate value theorem on Pareto-separating line in Edgeworth box
  - Intermediate value theorem on shape of offer curves in Edgeworth box
  - General Kakutani's fixed-point theorem argument
- There are a finite, odd number of W.E., each of which is locally unique (generically)
- It's a "hard" question how the economy finds W.E. prices dynamically; some W.E. are unstable (and tatonnement need not converge at all!)

Firms with constant returns to scale technology

Suppose a firm has CRS production technology; i.e.,
$y \in Y \implies \beta y \in Y$ for all y and all $\beta > 0$

What can we say about its profit?

- Can it be strictly positive? No. . . otherwise it could scale up production arbitrarily and achieve infinite profit
- Could be zero due to prices
- Could be zero due to shutdown

So ownership structure $(\alpha^{ki})_{ki}$ doesn't matter

Part XII

Appendix

Multivariate inequalities and orthants of Euclidean space

This notation can be tricky, but is often used carefully:

1. $\mathbb{R}^n_+ \equiv \{\mathbf{x} \colon \mathbf{x} \geq \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i \geq 0 \text{ for all } i\}$
   Includes the axes and $\mathbf{0}$

2. $\{\mathbf{x} \colon \mathbf{x} > \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i \geq 0 \text{ for all } i\} \setminus \mathbf{0}$
   Includes the axes, but not $\mathbf{0}$

3. $\mathbb{R}^n_{++} \equiv \{\mathbf{x} \colon \mathbf{x} \gg \mathbf{0}\} \equiv \{\mathbf{x} \colon x_i > 0 \text{ for all } i\}$
   Includes neither the axes nor $\mathbf{0}$

Separating Hyperplane Theorem I

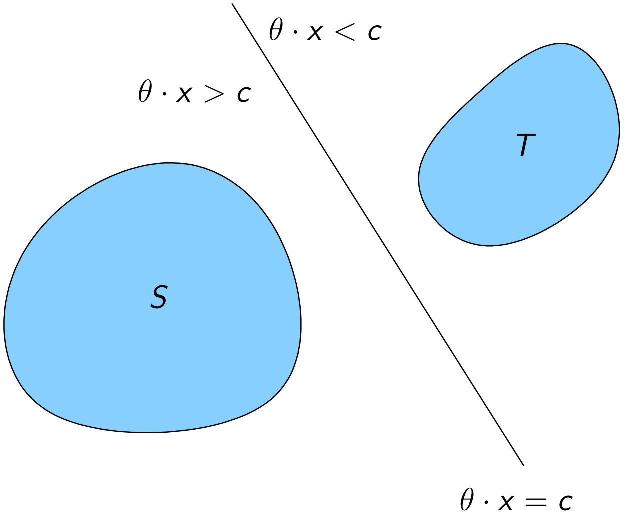
Theorem (Separating Hyperplane Theorem)

Suppose that S and T are two convex, closed, and disjoint ($S \cap T = \varnothing$) subsets of $\mathbb{R}^n$. Then there exists $\theta \in \mathbb{R}^n$ and $c \in \mathbb{R}$ with $\theta \neq \mathbf{0}$ such that

$$\theta \cdot s \geq c \text{ for all } s \in S \text{ and } \theta \cdot t < c \text{ for all } t \in T.$$

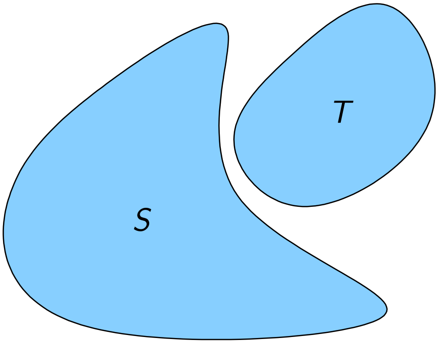
Means that a convex, closed set can be separated from any point outside the set

Separating Hyperplane Theorem II



Separating Hyperplane Theorem III

We can't necessarily separate nonconvex sets:



Convex functions

Definition (convexity)

$f : \mathbb{R}^n \to \mathbb{R}$ is convex iff for all x and $y \in \mathbb{R}^n$, and all $\lambda \in [0, 1]$, we have

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y).$$

Also characterized by $\mathbb{E}_G \left[f(x)\right] \geq f\left[\mathbb{E}_G(x)\right]$ for all distributions G
In the differentiable case, also characterized by any of

- If $f : \mathbb{R} \to \mathbb{R}$, then $f''(x) \geq 0$ for all x
- Hessian $\nabla^2 f(x)$ is a positive semidefinite matrix for all x
- $f(\cdot)$ lies above its tangent hyperplanes:

$$f(x) \geq f(y) + \nabla f(y) \cdot (x - y) \text{ for all } x \text{ and } y$$

Homogeneity and Euler's Law I

Definition (homogeneity)

$f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k iff for all $x \in \mathbb{R}^n$, and all $\lambda > 0$, we have

$$f(\lambda x) = \lambda^k f(x).$$

Theorem (Euler's Law)

Suppose $f(\cdot)$ is differentiable. Then it is homogeneous of degree k iff $p \cdot \nabla f(p) = kf(p)$.

Proof.

Homogeneous $\Rightarrow p \cdot \nabla f(p) = kf(p)$ proved by differentiating $f(\lambda p) = \lambda^k f(p)$ with respect to λ, and then setting $\lambda = 1$.
Homogeneous $\Leftarrow p \cdot \nabla f(p) = kf(p)$ may be covered in section. □

Homogeneity and Euler's Law II

Corollary

If $f(\cdot)$ is homogeneous of degree one, then $\nabla f(\cdot)$ is homogeneous of degree zero.

Proof.

Homogeneity of degree one means

$$\lambda f(p) = f(\lambda p).$$

Differentiating in p,

$$\begin{aligned} \lambda \nabla f(p) &= \lambda \nabla f(\lambda p) \\ \nabla f(p) &= \nabla f(\lambda p) \end{aligned}$$

□

The Kuhn-Tucker algorithm I

The Kuhn-Tucker Theorem provide the key generalization of the "Lagrangian" method for constrained optimization

Consider the problem

$$v(\theta) = \max_{x \in \mathbb{R}^n} f(x, \theta)$$

subject to constraints

$$g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0.$$

Set up a Lagrangian

$$\mathcal{L}(x, \theta, \lambda) \equiv f(x, \theta) + \sum_{k=1}^{K} \lambda_k g_k(x, \theta)$$

The Kuhn-Tucker algorithm II

Theorem (Kuhn-Tucker)

Suppose x^* solves the optimization problem at parameter θ, and

- $f(\cdot, \theta)$ *and* $g_1(\cdot, \theta), \ldots, g_K(\cdot, \theta)$ *are all differentiable in* x*;*
- *the constraint set is non-empty; and*
- *constraint qualification holds.*

Then there exist nonnegative $\lambda_1, \ldots, \lambda_K$ *such that*

1. *first-order conditions hold:*

$$D_x f(x^*, \theta) + \sum_{k=1}^{K} \lambda_k D_x g_k(x^*, \theta) = D_x \mathcal{L}(x^*, \lambda, \theta) = 0;$$

2. $\lambda_k g_k(x^*, \theta) = 0$ *(complementary slackness); and*
3. $g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0$ *(original constraints).*

The Kuhn-Tucker algorithm III

Kuhn-Tucker conditions are necessary and sufficient for a solution (assuming differentiability) as long as we have a “convex problem”:

1. The constraint set is convex
   - If each constraint gives a convex set, the intersection is a convex set
   - The set $\{x \colon g_k(x, \theta) \geq 0\}$ is convex as long as $g_k(\cdot, \theta)$ is a quasiconcave function of x
2. The objective function is concave
   - If we only know the objective is quasiconcave, there are other conditions that ensure Kuhn-Tucker is sufficient

Envelope Theorem I

ETs relate objective and value functions; this one relates the derivatives of objective and value functions for smooth PCOP:

Theorem (Envelope Theorem)

Consider a constrained optimization problem $v(\theta) = \max_x f(x, \theta)$ *such that* $g_1(x, \theta) \geq 0, \ldots, g_K(x, \theta) \geq 0$.

Comparative statics on the value function are given by:

$$\frac{\partial v}{\partial \theta_i} = \left.\frac{\partial f}{\partial \theta_i}\right|_{x^*} + \sum_{k=1}^{K} \lambda_k \left.\frac{\partial g_k}{\partial \theta_i}\right|_{x^*} = \left.\frac{\partial \mathcal{L}}{\partial \theta_i}\right|_{x^*}$$

(for Lagrangian $\mathcal{L}(x, \theta, \lambda) \equiv f(x, \theta) + \sum_k \lambda_k g_k(x, \theta)$*) for all* θ *such that the set of binding constraints does not change in an open neighborhood.*

Envelope Theorem II

The proof is given for a single constraint (but is similar for K constraints): $v(x,\theta) = \max_x f(x,\theta)$ such that $g(x,\theta) \geq 0$

Proof.

Lagrangian $\mathcal{L}(x,\theta) \equiv f(x,\theta) + \lambda g(x,\theta)$ gives FOC

$$\left.\frac{\partial f}{\partial x}\right|_* + \lambda \left.\frac{\partial g}{\partial x}\right|_* = \mathbf{0} \iff \left.\frac{\partial f}{\partial x}\right|_* = -\lambda \left.\frac{\partial g}{\partial x}\right|_* \tag{3}$$

where the notation $\cdot|_*$ means "evaluated at $\left(x^*(\theta),\theta\right)$ for some θ."
If $g\left(x^*(\theta),\theta\right) = 0$, take the derivative in θ of this equality condition to get

$$\left.\frac{\partial g}{\partial x}\right|_* \left.\frac{\partial x^*}{\partial \theta}\right|_\theta + \left.\frac{\partial g}{\partial \theta}\right|_* = \mathbf{0} \iff \left.\frac{\partial g}{\partial \theta}\right|_* = -\left.\frac{\partial g}{\partial x}\right|_* \left.\frac{\partial x^*}{\partial \theta}\right|_\theta . \tag{4}$$

Envelope Theorem III

Proof (continued).

Note that, $\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial f}{\partial \theta} + \lambda \frac{\partial g}{\partial \theta}$. Evaluating at $(x^*(\theta), \theta)$ gives

$$\left.\frac{\partial \mathcal{L}}{\partial \theta}\right|_* = \left.\frac{\partial f}{\partial \theta}\right|_* + \lambda \left.\frac{\partial g}{\partial \theta}\right|_\theta .$$

If $\lambda = 0$, this gives that $\left.\frac{\partial \mathcal{L}}{\partial \theta}\right|_* = \left.\frac{\partial f}{\partial \theta}\right|_*$; if $\lambda > 0$, complementary slackness ensures $g\big(x^*(\theta), \theta\big) = 0$ so we can apply equation 4. In either case, we get that

$$= \frac{\partial f}{\partial \theta} - \lambda \left.\frac{\partial g}{\partial x}\right|_* \left.\frac{\partial x^*}{\partial \theta}\right|_\theta . \qquad (5)$$

Envelope Theorem IV

Proof (continued).

Applying the chain rule to $v(x,\theta) = f\big(x^*(\theta),\theta\big)$ and evaluating at $\big(x^*(\theta),\theta\big)$ gives

$$\begin{aligned} \left.\frac{\partial v}{\partial \theta}\right|_* &= \left.\frac{\partial f}{\partial x}\right|_* \left.\frac{\partial x^*}{\partial \theta}\right|_\theta + \left.\frac{\partial f}{\partial \theta}\right|_* \\ &= -\lambda \left.\frac{\partial g}{\partial x}\right|_* \left.\frac{\partial x^*}{\partial \theta}\right|_\theta + \left.\frac{\partial f}{\partial \theta}\right|_* = \left.\frac{\partial \mathcal{L}}{\partial \theta}\right|_*, \end{aligned}$$

where the last two equalities obtain by equations 3 and 5, respectively. □

The Implicit Function Theorem I

A simple, general maximization problem

$$X^*(t) = \operatorname*{argmax}_{x \in X} F(x, t)$$

where $F \colon X \times T \to \mathbb{R}$ and $X \times T \subseteq \mathbb{R}^2$.

Suppose:

1. Smoothness: F is twice continuously differentiable
2. Convex choice set: X is convex
3. Strictly concave objective (in choice variable): $F''_{xx} < 0$ (together with convexity of X, this ensures a unique maximizer)
4. Interiority: $x(t)$ is in the interior of X for all t (which means the standard FOC must hold)

The Implicit Function Theorem II

The first-order condition says the unique maximizer satisfies

$$F_x\big(x(t), t\big) = 0$$

Taking the derivative in t:

$$x'(t) = -\frac{F_{xt}\big(x(t), t\big)}{F_{xx}\big(x(t), t\big)}$$

Note by strict concavity, the denominator is negative, so $x'(t)$ and the cross-partial $F''_{xt}\big(x(t), t\big)$ have the same sign

The Implicit Function Theorem: Higher dimensions

A more general general maximization problem

$X^*(t) = \text{argmax}_{x \in X} F(x, t)$ where $F\colon X \times T \to \mathbb{R}$ and $X \times T \subseteq \mathbb{R}^n$.

Under certain assumptions, we can apply a FOC:

$$\nabla_x F\big(x(t), t\big) = \mathbf{0}$$

Taking a derivative in t we get

$$\mathbf{0} = \frac{\partial^2 F\big(x(t), t\big)}{\partial x\, \partial x} \cdot \frac{\partial x(t)}{\partial t} + \frac{\partial^2 F\big(x(t), t\big)}{\partial x\, \partial t}$$

$$\frac{\partial x(t)}{\partial t} = -\left[\frac{\partial^2 F\big(x(t), t\big)}{\partial x\, \partial x}\right]^{-1} \cdot \frac{\partial^2 F\big(x(t), t\big)}{\partial x\, \partial t}$$

Multivariate Topkis' Theorem

Theorem (Topkis' Theorem)

Suppose

1. $F : X \times T \to \mathbb{R}$ *(for X a lattice, T partially ordered)*
   - *is supermodular in x (i.e., ID in all (x_i, x_j))*
   - *has ID in (x, t) (i.e., ID in all (x_i, t_j))*
2. $t' > t$,
3. $x \in X^*(t) \equiv \text{argmax}_{\xi \in X} F(\xi, t)$, *and* $x' \in X^*(t')$.

Then

$$(x \wedge x') \in X^*(t) \text{ and } (x \vee x') \in X^*(t').$$

That is, $X^*(\cdot)$ is nondecreasing in t in the stronger set order.

Quasiconcavity and quasiconvexity

Definition (quasiconcavity)

$f: X \to \mathbb{R}$ is quasiconcave iff for all $x \in X$, the upper contour set of x

$$\text{UCS}(x) \equiv \{\xi \in X : f(\xi) \geq f(x)\}$$

is a convex sets; i.e., if $f(\xi_1) \geq f(x)$ and $f(\xi_2) \geq f(x)$, then $f\big(\lambda\xi_1 + (1-\lambda)\xi_2\big) \geq f(x)$ for all $\lambda \in [0,1]$.

$f(\cdot)$ is strictly quasiconcave iff for all $x \in X$, UCS(x) is a strictly convex set; i.e., if $f(\xi_1) \geq f(x)$ and $f(\xi_2) \geq f(x)$, then $\lambda\xi_1 + (1-\lambda)\xi_2$ is an interior point of UCS(x) for all $\lambda \in (0,1)$.

Quasiconvexity and strict quasiconvexity replace "upper contour sets" with "lower contour sets" in the above definitions, where

$$\text{LCS}(x) \equiv \{\xi \in X : f(\xi) \leq f(x)\}$$

Why concavity implies quasiconcavity I

Theorem

A concave function is quasiconcave. A convex function is quasiconvex.

Note that showing a function is quasiconcave/quasiconvex is often harder than showing it is concave/convex

Why concavity implies quasiconcavity II

Proof.

Showing that concavity implies quasiconcavity is equivalent to showing that non-quasiconcavity implies non-concavity.

Suppose $f\colon X \to \mathbb{R}$ is not quasiconcave; i.e., there exists some x such that the upper contour set of x

$$\mathrm{UCS}(x) \equiv \{\xi \in X\colon f(\xi) \geq f(x)\}$$

is not a convex set.

Why concavity implies quasiconcavity III

Proof (continued).

For UCS(x) to be nonconvex, there must exist some x_1, $x_2 \in$ UCS(x) and $\lambda \in [0, 1]$ such that $\lambda x_1 + (1 - \lambda)x_2 \notin$ UCS(x); that is

$$\begin{aligned} f(x_1) &\geq f(x), \\ f(x_2) &\geq f(x), \\ f\big(\lambda x_1 + (1 - \lambda)x_2\big) &< f(x). \end{aligned}$$

By the above inequalities,

$$\lambda f(x_1) + (1 - \lambda)f(x_2) > f\big(\lambda x_1 + (1 - \lambda)x_2\big),$$

and $f(\cdot)$ is therefore not concave. □